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ERIC DAL MORO[*], YURIY KRVAVYCH[†]

The upcoming Solvency II and IFRS 4 Phase II regimes bring significant changes to current reporting of insurance entities, and particularly in relation to valuation of insurance liabilities. Insurers will be required to valuate their insurance liabilities on a risk-adjusted basis to allow for uncertainty inherent in cash flows that arise from the liability of insurance contracts. Whilst most European-based insurers are expected to adopt the Cost of Capital (CoC) approach to calculating reserve risk margin - the risk adjustment method commonly agreed under Solvency II and IFRS 4 Phase II, there is one additional requirement of IFRS to also disclose confidence level of the risk margin.

Given there is no specific guidance on the calculation of confidence level, the purpose of this paper is to examine the implied level of confidence/prudence, measured by Probability of Sufficiency (PoS), of reserve risk margins under Solvency II CoC approach to valuating insurance liabilities. The paper provides some practical approximation formulae that would allow one to quickly estimate the implied PoS of Solvency II risk margin for a given non-life insurance liability, the risk profile of which is specified by the type and characteristics of the liability (e.g. type/nature of business, liability duration and convexity, etc.), which in turn are associated with

- the level of variability measured by Coefficient of Variation (CoV);
- the degree of Skewness per unit of CoV; and
- the degree of Kurtosis per unit of CoV^2.

The approximation formulae of PoS are derived for both the standalone class risk margin and the diversified risk margin at the portfolio level.

Keywords: IFRS 4 Phase II; IFRS confidence level of Solvency II risk margins; Cost of Capital approach; Probability of Sufficiency; Approximations

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1 INTRODUCTION

Insurance liabilities are generally evaluated using the market-consistent valuation under which the liabilities could be transferred to a willing rational counterparty at a price that would fully reflect the market/buyer's perception of risk. The price for assuming risk from insurance liability portfolio transfer comes in the form of a *Risk Margin* that stacks on top of the *Central Estimate*[1] of the liabilities.

As there is no established liquid market for trading insurance liabilities, the market-consistent value of liability cannot be deduced directly from the capital market, and hence various market-consistent valuation techniques are usually used for internal valuation of non-hedgeable insurance liabilities. Examples of such market-consistent valuation techniques include *Replicating Portfolio*, *Valuation Portfolio* and *Cost of Capital (CoC)*. The former two employ ideas of financial mathematics and are commonly used in the valuation of liabilities in life insurance and pensions (cf. Møller and Steffensen [15], Bühlmann [2], and Wüthrich et al. [24]), whereas the latter directly associates the risk margin with the cost of servicing the economic capital attracted by the acquisition/transfer of insurance liability portfolio. The CoC approach is the main market-consistent valuation approach to calculating risk margin for insurance liabilities under Solvency II, and is particularly prescribed for valuation of non-hedgeable liabilities in non-life insurance[2].

The CoC approach is not a new concept and has been in use in actuarial practice in various forms mainly for setting profit margins in premium rates and to a certain extent for reporting embedded value. It also resonates with some traditional approaches to allowing for reserve uncertainty/variability in liability valuation. For example, in Australia the non-life technical provisions are required by APRA[3] to be set with the minimum of 75% of Probability of Sufficiency (PoS), in which case the implied risk margin is the difference between the 75th percentile of liability distribution profile and the central estimate. In general, the PoS itself is a measure of prudence in liability valuation:

- PoS below 50% indicates the technical provisions are set below the central estimate (under-reserved position);
- PoS of 50% to 60% indicates the technical provisions are set approximately at the level of central estimate (weak prudence);
- PoS of around 75% indicates that technical provisions are set so that likely (i.e. up to 1-in-4 years) reserve deteriorations above the central estimate are fully absorbed by the technical provisions (adequate prudence); and
- PoS above 75% indicates that the technical provisions could also absorb some of unlikely reserve deteriorations above the central estimate (strong prudence).

Under the upcoming Solvency II and IFRS 4 Phase II regimes insurers will be required to valuate their insurance liabilities on a risk-adjusted basis to allow for uncertainty inherent in cash flows that arise from the liability of insurance contracts. Whilst most European-based insurers are expected to adopt the CoC approach to calculating reserve risk margin - the risk adjustment method commonly agreed under Solvency II and IFRS 4 Phase II, there is one additional requirement of IFRS to also disclose confidence level of the risk margin.

1 In general, this could be *mean*, *mode* or *median*. However, unbiased mean is usually used to evaluate the Central Estimate of liabilities.

2 From the financial theory point of view shareholders of insurance undertaking provide capital to support the acquisition/transfer of the insurance liability portfolio, and thus would require reward for the capital supplied to support the non-hedgeable risk.

3 Australian Prudential Regulation Authority.

Given that there is no specific guidance on the calculation of confidence level, the purpose of this paper is to examine the implied level of confidence/ prudence, measured by PoS, of reserve risk margins under Solvency II CoC approach to valuating insurance liabilities. The paper provides some practical approximation formulae that would allow one to quickly estimate the implied PoS of Solvency II risk margin for a given non-life insurance liability, the risk profile of which is specified by the type and characteristics of the liability (e.g. type/nature of business, liability duration and convexity, etc.), which in turn are associated with

- the level of variability measured by Coefficient of Variation (CoV);
- the degree of Skewness per unit of CoV, i.e. Skewness-to-CoV (SC) ratio; and
- the degree of Kurtosis per unit of CoV^2, i.e. Kurtosis-to-CoV^2 (KCsq) ratio.

The approximation formulae of PoS are derived for both the standalone class risk margin and the diversified risk margin at the portfolio level.

The structure of this paper is as follows. In section 2, we define and outline the notions of reserve risk, its profile and risk margin, and provide the key assumptions made in this research. In section 3, we focus on providing practical approximation formulae for PoS of CoC risk margin of a standalone reserving class. Here, the PoS, being an inverse-quantile measure of the risk profile of the single reserving class, is approximated using the *Cornish-Fisher* approximation (see Fisher and Cornish [7]). This is the quantile approximation of the non-normal distribution by a polynomial of a standard normal quantiles utilising the distribution's moments. The approximation formulae for PoS are derived separately for two different cases: 1) when utilising only coefficient of variation and skewness of the non-normal distribution; and 2) when utilising its coefficient of variation, skewness and kurtosis. The obtained approximations of PoS are compared by analysing their quality. In section 4, we provide approximations of PoS of CoC risk margin of a portfolio consisting of multiple reserving classes. In general, the derivation of the distribution of aggregate risk at the portfolio level is to a large extent associated with dependence modelling uncertainty (see, e.g., Embrechts and Jacobsons [6]). In this paper we consider a Gaussian dependence structure for aggregating risks[4] and estimate the first four moments of the aggregate reserve risk profile using the *Fleishman* approximation[5] (see Fleishman [8]). The derived moments are further used to approximate the PoS of the diversified risk margin at the portfolio level by utilising formulae obtained in section 3. Finally, brief conclusions are given in section 5.

2 Reserve risk profile and risk margin - background and general assumptions

2.1 *Reserve risk profile characteristics*

Reserve risk and its carrier. Under Solvency II the '*risk*' is generally defined as a possibility of having adverse performance result (insurance, investment, or company's overall result) that results in 'low capital performance' (i.e. a return on capital below the shareholders opportunity cost of capital) and/or

4 The idea of applying a Gaussian dependence structure in reserve risk aggregation was earlier employed in Dal Moro [3] and [4] when deriving the moments of aggregate reserve risk across a multivariate log-normal reserve risk profile. This paper applies a Gaussian dependence structure to any multivariate distribution and derives the four moments of the aggregate reserve risk profile by using Cornish-Fisher and Fleishman approximations.

5 Fleishman approximation is a polynomial of a standard normal variable allowing to approximate non-normal distributions by utilising their moments of up to the fourth order.

erosion of current shareholders value (i.e. capital consumptions)[6]. In particular, the reserve risk is the risk that provisions for past exposures will be inadequate to meet the ultimate costs when the business is run off to extinction. The risk of reserves developing other than expected (booked provisions) is significant for non-life insurers, especially for long tail lines of business. Here, the reserve provisions are generally booked at the *Central Estimate* plus *Risk Margin*, where the risk margin plays the role of safety load reflecting the uncertainty in reserve central estimate. The role of the risk margin is further explained in greater detail below in subsection 2.2

The risk is often distinguished from its carrier which is defined as a random variable. In the case of reserve risk its carrier is naturally defined as the *reserve value*, which is random due to random nature of claims frequency and severity and also random time lags between: a) the date the insurance event occurs and the date it is reported; and b) the date it is reported and the date it is eventually settled. Because these time lags, along with underlying claim frequencies and severities, are stochastic, booked claims liabilities have substantial risk that their actual value realised in the future will adversely deviate from the expected value (booked provisions).

Assumption 1. It is the distribution of the reserve risk carrier that characterises the reserve risk, and in this paper it is referred to as the *'reserve risk profile'*.

RESERVE RISK PROFILE: DIFFERENTIATION BY TYPE OF BUSINESS CLASS. In practice, non-life reserving actuaries often use *Coefficient of Variation* (CoV) as the measure of riskiness of modelled reserves. For example, personal lines like motor and home are short tail business lines and exhibit relatively lower CoV when compared to long tail classes like commercial liability. However, CoV alone cannot explain all the characteristics of the reserve risk profile, and thus higher moments of reserve distribution like skewness and kurtosis would be required to properly capture a) the degree of asymmetry of odds towards adverse reserve realisations, and b) heavy-tailedness of the reserve distribution.

In general, the parametric distributions commonly used in insurance for reserving and loss modelling are of a special type:

- they are often defined by two parameters - the *scale parameter* and the *shape parameter*; and
- their shape is totally driven by a single parameter - their shape parameter.

Equivalently, those two-parameter distributions are such, that when scaled by their mean (location), would have a unique fixed location (unit mean) and variable shape dependent on the shape parameter only, i.e. the distribution of the following random variable Y is a single-parameter distribution and its shape is defined by the shape parameter of X:

$$Y = \frac{1}{m_X} X = 1 + \mathrm{CoV}_X \cdot \widetilde{X}, \tag{1}$$

where $\widetilde{X} = \frac{X - m_X}{m_X \,\mathrm{CoV}_x}$ and m_X and CoV_X are the mean and CoV of X respectively.

In this paper, we focus only on the class of two-parameter distributions with *single shape parameter* and denote it by $\mathcal{SSP}$. Examples of two-parameter distributions of $\mathcal{SSP}$ type include[7] Gamma, Inverse-Gaussian (Wild), Log-Normal, Dagum, Suzuki, Exponentiated-Exponential (Verhulst), Inverse-Gamma (Vinci), Birnbaum-Saunders, Exponentiated-Fréchet and Log-Logistic. It should be

6 E.g., see Krvavych [12]

7 Please refer to Kleiber and Kotz [11], Nadarajah and Kotz [17], Nadarajah [16], Marshall and Olkin [14] and Wolfram Documentation Center [23]

noted, that not all two-parameter distributions are of $\mathcal{SSP}$ type, and immediate example of that would be the Log-Gamma distribution each of the two parameters of which would drive both the location/scale and the shape of the distribution at the same time.

As was illustrated above in (1), for any distribution of $\mathcal{SSP}$ type its shape in general and its CoV in particular are defined by the distribution's shape parameter only. Also, it is known fact from the Distribution Analysis of the Probability Theory that any analytical cumulative distribution function can be expanded using the Cornish-Fisher expansion (cf. Fisher and Cornish [7]), which utilises the distribution's skewness, kurtosis[8] and other relative moments of higher order to fully explain its shape. In the case of random variable $\widetilde{X}$, the shape of its distribution is completely explained by the skewness, kurtosis and other relative moments of higher order of X. This implies that all relative moments of third order and higher of any distribution of $\mathcal{SSP}$ type are completely defined by its shape parameter.

Therefore, the distinctive features of the distributions of $\mathcal{SSP}$ type are:

- their shape parameter is a function of CoV;
- their any higher-order statistic is fully determined by the shape parameter and hence is a function of CoV, and in particular
  - relative skewness measured by Skewness-to-CoV (SC) ratio is a function of CoV;
  - relative kurtosis measured by Kurtosis-to-CoV^2 (KCsq) ratio is a function of CoV^2;
- if the $\mathcal{SSP}$ distribution belongs to a certain parametric family (e.g. Log-Normal, Gamma or any other parametric family from $\mathcal{SSP}$) then scaling it by its mean preserves the parametric family it belongs to and its shape parameter, i.e. if $X \sim F_{u,v}$ with mean m_X and standard deviation s_X, scale parameter u and shape parameter v, then $Y = \frac{1}{m_X}X \sim F_{u',v}$ with mean 1 and standard deviation CoV_X.

The latter feature also implies that

$$F_{u,v}(x) = F_{u',v}\left(\frac{x}{m_X}\right). \tag{2}$$

The $\mathcal{SSP}$ distributions can be split into three main categories:

- **Moderately skewed distributions** ($1.5 < SC \leq 3$)
  - *Gamma*;
  - *Inverse-Gaussian (Wald)*;
- **Significantly skewed distributions** ($3 < SC < 4$)
  - *Log-Normal*;
  - *Suzuki*;
  - *Exponentiated-Exponential (Verhulst)*;
  - *Dagum*;
- **Extremely skewed distributions** ($4 < SC < 5.5$)
  - *Inverse-Gamma (Vinci)*;
  - *Birnbaum-Saunders*;
  - *Log-Logistic*;

8 Here, kurtosis is regarded as excess-kurtosis and thus is defined via the fourth- and second-order cumulants of the reserve distribution.

– *Exponentiated-Fréchet.*

The parametric distribution that is most commonly used in reserving is the Log-Normal distribution. Its skewness and kurtosis are respectively defined by corresponding ratios:

$$SC = 3 + CoV^2 > 3, \tag{3}$$
$$KCsq = 16 + 15CoV^2 + 6CoV^4 + CoV^6 > 16 \tag{4}$$

The Log-Normal distribution, whilst being suitable for modelling a wide range of skewed medium- to heavy-tailed risk profiles, is still not the best one for modelling risk profiles with low skewness and light tail or excessively high skewness and heavy tail.

Assumption 2. This paper considers the following four parametric distributions that could in principle cover the whole range of the reserve risk profiles used in practice:

- *Gamma* – for modelling reserve risk profiles with relatively low skewness and light tail;
- *Inverse-Gaussian (Wald)* – for modelling reserve risk profiles with moderate skewness and heavy-tailedness;
- *Log-Normal* – for modelling reserve risk profiles with medium to large skewness and heavy-tailedness;
- *Inverse-Gamma (Vinci)* – for modelling reserve risk profiles with excessively large skewness and heavy-tailedness,

and assumes that any reserve risk profile with $SC = k_S \times CoV$ and $KCsq = k_K \times CoV^2$ for a given level of CoV (k_S and k_K are positive multipliers) could be associated with the closest distribution curve out of the four parametric distributions considered. We denote the set of the four proposed parametric distributions by $\mathcal{PD} \subset \mathcal{SSP}$. This is the key assumption of reserve risk profile characterisation that is further used in the derivation of PoS approximation formulae in section 3 and section 4.

We further use the $\mathcal{PD}$ set of parametric distributions to illustrate how reserve risk profiles could be differentiated by type of business/reserving class. This is presented in Table 1.

Table 1: *Differentiation of reserve risk profile by type of reserve class.*

Type of reserving class				
Duration	**CoV range**	**Skewness (SC ratio)**	**Parametric distribution(s)**	**Example of reserving class**
Short tail	10%-12%	1.9 to 2.1	Gamma	Motor (ex Bodily Injury)
Short tail	12%-16%	2.0 to 3.0	Gamma, Inverse-Gaussian (Wald)	Home
Short tail	10%-16%	2.9 to 3.1	Inverse-Gaussian (Wald), Log-Normal	Comm Property/Fire, Comm Accident
Long tail	12%-25%	3.0 to 3.5	Log-Normal	Motor Bodily Injury, Marine
Long tail	18%-50%	3.0 to 4.0	Log-Normal, Inverse-Gamma (Vinci)	Workers Comp, Prof Liab, Comm Liab
Long tail	25%-70%	> 4	Inverse-Gamma (Vinci)	Asbestos and other long tail books

The characteristics of the four proposed parametric distributions are provided in Table 2 below.

The graphs of Skewness and Kurtosis as functions of CoV derived from SC and $KCsq$ ratios are provided below in Figure 1 and Figure 2.

Table 2: *SC and KCsq ratios for the four parametric distributions.*

Parametric distribution	SC ratio as a function of CoV	KCsq ratio as a function of CoV^2
Gamma	2	6
Inverse-Gaussian (Wald)	3	15
Log-Normal	$3 + CoV^2 \in (3, 4),\ CoV < 100\%$	$16 + 15CoV^2 + 6CoV^4 + CoV^6 > 16$
Inverse-Gamma (Vinci)	$\frac{4}{1-CoV^2} > 4,\ CoV < 100\%$	$\frac{30\left(1-\frac{1}{5}CoV^2\right)}{(1-CoV^2)(1-2CoV^2)} > 30,\ CoV < 70\%$

Figure 1: *Skewness as a function of CoV for the four parametric distributions.*

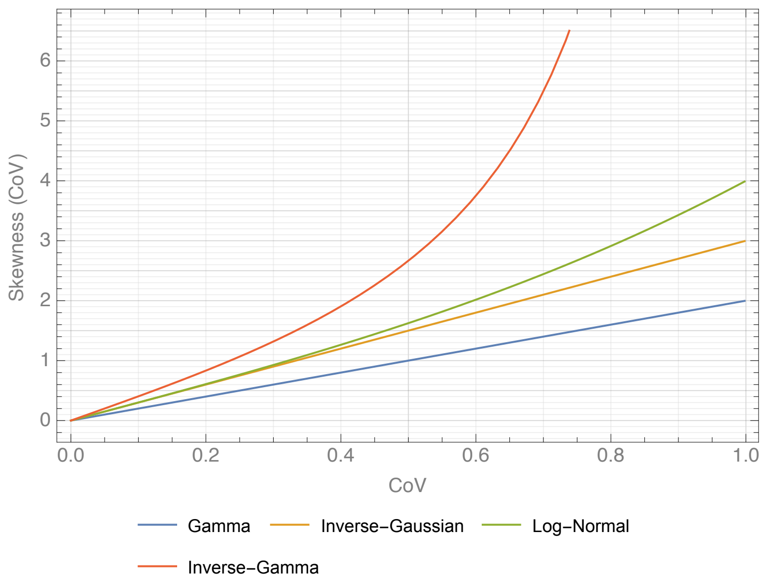
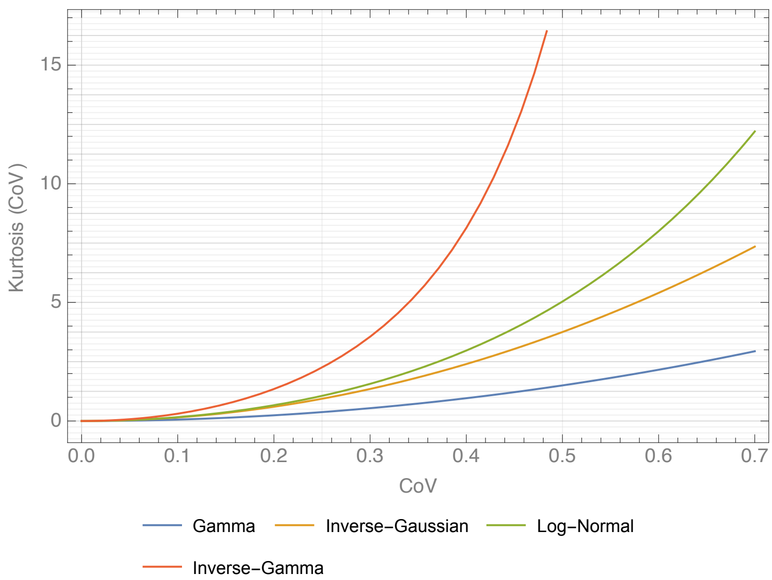


Figure 2: *Kurtosis as a function of CoV for the four parametric distributions.*



These graphs demonstrate monotonic increase in the level of Skewness and Kurtosis for a given level of CoV when moving sequentially across the set $\mathcal{PD}$ of the proposed four parametric distributions from Gamma to Inverse-Gamma.

AGGREGATE RESERVE RISK PROFILE. The derivation of the distribution of aggregate risk at the portfolio level is in general not a trivial task and to a large extent is associated with dependence modelling uncertainty. In Embrechts and Jacobsons [6] the authors studied the lower and upper bounds of the quantile estimate of the aggregate risk distribution and showed how wide the interval estimate could be when the uncertainty in the choice of the most appropriate dependence structure is taken into account.

Assumption 3. The choice of the most appropriate dependence structure should be determined through the use of adequate calibration process. However, this paper does not focus on the calibration of the most appropriate dependence structure for aggregating reserve risk, but rather assumes that in practice the Gaussian dependence structure is likely to be reasonable, if not the most appropriate one, for aggregating reserve risk.

It is also assumed that the matrix of Kendall's tau correlation coefficients, R, is available and used to calibrate the Gaussian copula. This means that the rank correlation (Kendall tau) between classes i and j, $r_{ij} = R(i, j)$, is transformed to the corresponding linear correlation coefficient ρ_{ij} in the following way:

$$\rho_{ij} = \sin\left(\frac{\pi}{2} r_{ij}\right). \tag{5}$$

Assumption 4. It is assumed in this paper that the aggregate reserve risk profile is characterised sufficiently well by its moments of up to the fourth order, i.e. mean central estimate, coefficient of variation, skewness and kurtosis. In particular, the random reserve value, X, can be expressed through its centralised and normalised copy:

$$X = CE_X \cdot \left(1 + CoV_X \cdot \widetilde{X}\right), \tag{6}$$

where $\widetilde{X} = \frac{X - CE_X}{CE_X \cdot CoV_X}$ is a non-normal random variable with zero mean and unit standard deviation. The random variable $\widetilde{X}$ is then approximated using the Fleishman approximation:

$$\widetilde{X} = aZ + b(Z^2 - 1) + cZ^3, \quad Z \sim \mathcal{N}(0, 1), \tag{7}$$

where the Fleishman coefficients a, b and c are calibrated by matching the second, third and fourth moments of $\widetilde{X}$ to 1 (standard deviation), γ (skewness) and $\iota + 3$ (absolute or non-centralised kurtosis) respectively. Please note that γ and ι are also skewness and kurtosis of X as they are invariant[9] with respect to translation and scaling. In section 4, the paper considers two different cases of Fleishman approximation:

1. approximation using skewness only, i.e. $\widetilde{X} = aZ + b(Z^2 - 1)$; and
2. approximation using skewness and kurtosis, i.e. $\widetilde{X} = aZ + b(Z^2 - 1) + cZ^3$.

The calibration of Fleishman coefficients in the first approximation is straightforward and can be analytically expressed, whereas the coefficients of the second approximation are numerically pre-computed and tabulated in Appendices for a given level of CoV and the skewness and kurtosis expressed as functions of CoV.

9 Skewness and Kurtosis are respectively $\gamma(X) = \frac{\kappa_3(X)}{\kappa_2^{3/2}(X)}$ and $\iota(X) = \frac{\kappa_4(X)}{\kappa_2^2(X)}$ and are invariant with respect to centralisation and standardisation of X. Here, $\kappa_n(X)$ is the n-th cumulant of X.

These are the key assumptions used in the derivation of characteristics of aggregate reserve risk profile outlined in section 4. The work done in section 4 could be further extended to other types of copula structures. For example, the recent research work by Gijbels and Herrmann [9] provides some useful insight into the approximation of distribution of sum of random variables with copula-induced dependence structure. This can be utilised in order to alter the approximation formulae for PoS of risk margin at the aggregate reserve risk level already obtained in section 4 to allow for the use of other non-Gaussian dependence structures.

2.2 *Reserve risk margin and its level of confidence*

RISK MARGIN. The risk margin of the non-hedgeable insurance liabilities is somewhat similar to what is called *'available solvency margin'* (or capital requirements) under Solvency II. Both represent some form of safety load and are important and necessary to ensure the overall sufficiency of the solvency assessment. However, the role the risk margin plays is quite distinctive from that of 'available solvency margin'. The risk margin reflects the *uncertainty* in the central estimates of reserves, whereas the actual volatility of reserving process (reserve risk) is fully absorbed by capital requirements. This can be illustrated by decomposing the mean square error of estimating the unknown true central estimate of reserves, CE, by the best central estimate, $\widehat{CE}$, provided by a reserving actuary:

$$\begin{aligned}\mathbb{E}\left[\left(CE-\widehat{CE}\right)^2\right] &= \mathbb{E}\left[\left(\left(CE-\mathbb{E}\left[\widehat{CE}\right]\right)+\left(\mathbb{E}\left[\widehat{CE}\right]-\widehat{CE}\right)\right)^2\right]\\ &= \text{bias}^2+\text{Var}\left[\widehat{CE}\right]\\ &= \text{bias}^2+\mathbb{E}\left[\text{Var}\left[\widehat{CE}\,|\,\mathcal{F}\right]\right]+\text{Var}\left[\mathbb{E}\left[\widehat{CE}\,|\,\mathcal{F}\right]\right], \qquad (8)\end{aligned}$$

where $\mathcal{F}=\sigma\left\{\mathbf{M},\mathbf{CE}\right\}$ is the information filtration (σ-algebra) on a set of models $\mathbf{M}$ and a set of true parameters $\mathbf{CE}$. The first term in (8), mean squared bias, is a matter of insurance supervisor's attention and its minimisation is regulated in Pillar II of Solvency II. The second term is the expected variability of reserve risk process which is fully captured by the Solvency II capital requirements, and the latter term is the volatility of the best central estimate which represents the uncertainty in the true central estimate of reserve and the choice of the most appropriate actuarial model. And, therefore, this latter term is captured by the risk margin.

Under the upcoming Solvency II the prescribed approach to calculating risk margin of non-hedgeable insurance liabilities is the CoC approach. Under the CoC approach the margin for uncertainty in liability valuation is directly linked to the market price for assuming reserve risk from insurance liability portfolio transfer by a willing third party (cf. Strommen [22]). This market price is measured by the cost of regulatory capital required to runoff all liabilities. Exactly in such a context the CoC approach was first introduced in the Swiss Solvency Test (SST) in 2004 (cf. SST [21], SCOR [19] and Sandström [18]). The derivation of CoC risk margin entails

- projecting the Solvency Capital Requirements (SCR) net of diversification benefits for non-hedgeable risks from time $t=1$ to full runoff;
- calculating the capital charge at the end of each projection year by multiplying SCR_t by the CoC charge c_t; and
- taking the total present value of projected capital charges to determine the overall CoC risk margin.

Assumption 5. It is not the aim of this paper to provide the derivation of the CoC risk margin as it is assumed here that it is pre-calculated and provided by reserving and/or capital actuaries at both single class level and the portfolio level. The CoC risk margins are assumed to be expressed as a percentage of reserve central estimate CE and denoted by

- η_i – relative risk margin for i-th standalone reserving class; and
- η_Σ – relative risk margin for the portfolio of multiple reserving classes.

LEVEL OF CONFIDENCE: MEASUREMENT AND APPROXIMATION. The IFRS level of confidence of CoC risk margin is measured by Probability of Sufficiency (PoS). For a given random reserve value X with the central estimate CE_X, the level of variability CoV_X and the risk margin η_X the PoS is defined as

$$PoS = \mathbb{P}\left[X \leq CE_X \cdot (1 + \eta_X)\right] = \alpha. \tag{9}$$

To solve for unknown level of PoS, α, one would need first to invert PoS by taking Value at Risk (VaR) of X at α

$$\mathrm{VaR}_\alpha(X) = CE_X \cdot (1 + \eta_X), \tag{10}$$

and then express the $\mathrm{VaR}_\alpha(X)$ through the VaR of centralised and normalised copy of X, $\widetilde{X} = \frac{X - CE_X}{CE_X \cdot CoV_X}$, and solve the following equation for α

$$CE_X \cdot \left(1 + CoV_X \cdot \mathrm{VaR}_\alpha\left(\widetilde{X}\right)\right) = CE_X \cdot (1 + \eta_X), \tag{11}$$

or equivalently the equation

$$\mathrm{VaR}_\alpha\left(\widetilde{X}\right) = \frac{\eta_X}{CoV_X}. \tag{12}$$

It should be noted that the linearity of VaR transformation used in Equation 11 holds for any distribution.
The equation (12) indicates that the level of PoS is dependent only on the relative risk margin η_X (the risk margin per unit of CE_X) and the level of variability of X measured by CoV_X, and thus is invariant with respect to CE_X - the location of the risk profile of X.

Assumption 6. To solve the equation (12) this paper suggests using Cornish-Fisher approximation of $\mathrm{VaR}_\alpha\left(\widetilde{X}\right)$. The Cornish-Fisher approximation is derived from the Cornish-Fisher expansion series of the quantiles of a random variable via its cumulants and the standard normal quantiles. The detailed derivation of the Cornish-Fisher expansion can be found in Fisher and Cornish [7] and Lee and Lee [13]. This paper assumes that the $\mathrm{VaR}_\alpha\left(\widetilde{X}\right)$ can be well approximated by the moments of X of up to fourth order. In particular, the following three variants of Cornish-Fisher approximations are considered:

1. second-order Cornish-Fisher (or, equivalently, first-order Normal Power) approximation utilising only skewness γ_X of X:

$$\mathrm{VaR}_\alpha(\widetilde{X}) \approx z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6}, \tag{13}$$

   where z_α is the standard normal quantile at probability level α;

2. third-order Cornish-Fisher (or, equivalently, second-order Normal Power) approximation utilising skewness γ_X and kurtosis ι_X of X:

$$\begin{aligned} \mathrm{VaR}_\alpha(\widetilde{X}) \approx\ & z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6} + \iota_X \frac{z_\alpha^3 - 3z_\alpha}{24} \\ & - \gamma_X^2 \frac{2z_\alpha^3 - 5z_\alpha}{36}; \end{aligned} \tag{14}$$

3. fourth-order Cornish-Fisher approximation utilising skewness γ_X and kurtosis ι_X of X:

$$\begin{aligned}\mathrm{VaR}_\alpha(\widetilde{X}) \quad \approx \quad & z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6} + \iota_X \frac{z_\alpha^3 - 3z_\alpha}{24} \\ & -\gamma_X^2 \frac{2z_\alpha^3 - 5z_\alpha}{36} - \gamma_X \iota_X \frac{z_\alpha^4 - 5z_\alpha^2 + 2}{24} \\ & +\gamma_X^3 \frac{12z_\alpha^4 - 53z_\alpha^2 + 17}{324}. \end{aligned} \tag{15}$$

We denote the set of the three proposed PoS approximations by $\mathcal{A}$. To find unknown α from (12), one would need first to find the unknown standard normal quantile z_α from (13), (14) and (15) by solving their corresponding *quadratic*, *cubic* and *quartic* equations, and then inverting it to α by taking $\Phi(z_\alpha)$.

3 POS APPROXIMATIONS FOR COC RISK MARGIN - STANDALONE RESERVING CLASS

This section focuses on providing approximation formulae for PoS level of CoC reserve risk margin of a standalone reserving class. It considers three different variants of Cornish-Fisher approximations of the VaR of centralised and normalised reserve value that drives the PoS level of risk margin:

1. second-order Cornish-Fisher (or first-order Normal Power) approximation;
2. third-order Cornish-Fisher (or second-order Normal Power) approximation; and
3. fourth-order Cornish-Fisher approximation.

The quality of obtained three approximations is analysed.

3.1 *First-order Normal Power approximation*

The first-order Normal Power approximation of $\mathrm{VaR}_\alpha\left(\widetilde{X}\right)$ utilises only skewness γ_X of X:

$$\mathrm{VaR}_\alpha(\widetilde{X}) \quad \approx \quad z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6}. \tag{16}$$

Let $q = \frac{\eta_X}{CoV_X}$. Then, to find unknown z_α we solve the quadratic equation

$$\begin{gathered} z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6} = q, \quad \text{or equivalently} \\ z_\alpha^2 + \frac{6}{\gamma_X} z_\alpha - \left(\frac{6}{\gamma_X} q + 1\right) = 0. \end{gathered} \tag{17}$$

There is only one real positive root of the quadratic equation (17) and it is equal to

$$\widehat{z}_\alpha = -\frac{3}{\gamma_X} + \sqrt{\frac{9}{\gamma_X^2} + \frac{6}{\gamma_X} q + 1}. \tag{18}$$

The PoS level, α, is then found as follows

$$\widehat{\alpha} = \Phi(\widehat{z}_\alpha). \tag{19}$$

3.2 *Second-order Normal Power approximation*

The second-order Normal Power approximation of $\mathrm{VaR}_\alpha\left(\widetilde{X}\right)$ utilises both skewness γ_X and kurtosis ι_X of X:

$$\begin{aligned}\mathrm{VaR}_\alpha(\widetilde{X}) &\approx z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6} + \iota_X \frac{z_\alpha^3 - 3z_\alpha}{24} \\ &\quad -\gamma_X^2 \frac{2z_\alpha^3 - 5z_\alpha}{36}.\end{aligned} \tag{20}$$

To find unknown z_α, we solve the following cubic equation

$$z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6} + \iota_X \frac{z_\alpha^3 - 3z_\alpha}{24} - \gamma_X^2 \frac{2z_\alpha^3 - 5z_\alpha}{36} = q,$$

or equivalently

$$a z_\alpha^3 + b z_\alpha^2 + c z_\alpha + d = 0, \tag{21}$$

where

$$\begin{cases} a = \frac{1}{24}\iota_X - \frac{1}{18}\gamma_X^2, \\ b = \frac{1}{6}\gamma_X, \\ c = 1 + \frac{5}{36}\gamma_X^2 - \frac{1}{8}\iota_X, \\ d = -\frac{1}{6}\gamma_X - q. \end{cases} \tag{22}$$

The roots of the cubic equation (21) can be found using the Cardano's formula (see, e.g., Abramowitz and Stegun [1])

$$\begin{cases} x_1 = M + N - \frac{b}{3a}, \\ x_2 = -\frac{M+N}{2} - \frac{b}{3a} + \frac{i\sqrt{3}}{2}(M - N), \\ x_3 = -\frac{M+N}{2} - \frac{b}{3a} - \frac{i\sqrt{3}}{2}(M - N), \end{cases} \tag{23}$$

where

$$M = \sqrt[3]{P + \sqrt{P^2 + Q^3}}, \tag{24}$$

$$N = \sqrt[3]{P - \sqrt{P^2 + Q^3}}, \tag{25}$$

where

$$P = \frac{9abc - 27a^2 d - 2b^3}{54a^3}, \tag{26}$$

$$Q = \frac{3ac - b^2}{9a^2}. \tag{27}$$

The existence of real roots of the cubic equation (21) and their quantity are dependent on the sign of cubic discriminant

$$D = P^2 + Q^3. \tag{28}$$

We consider three cases of D:

1. if $D > 0$, then there exists only one real root and it is equal to $M + N - \frac{b}{3a}$;
2. if $D = 0$, then all three roots are real, and at least two are the same and equal to $-\frac{M+N}{2} - \frac{b}{3a}$; and
3. if $D < 0$, then all three roots are real and unequal.

In the latter case, when $D < 0$, the three real roots can also be expressed trigonometrically:

$$\begin{cases} x_1 &= 2\sqrt{-Q}\cos\left(\frac{\varphi}{3}\right) - \frac{b}{3a}, \\ x_2 &= 2\sqrt{-Q}\cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) - \frac{b}{3a}, \\ x_3 &= 2\sqrt{-Q}\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) - \frac{b}{3a}, \end{cases} \tag{29}$$

where $\varphi = \arccos\left(\frac{P}{\sqrt{-Q^3}}\right)$.

In any case, the root $\widehat{z}_\alpha$ is then the largest of all real roots of the cubic equation (21). The PoS level, α, is then found as follows

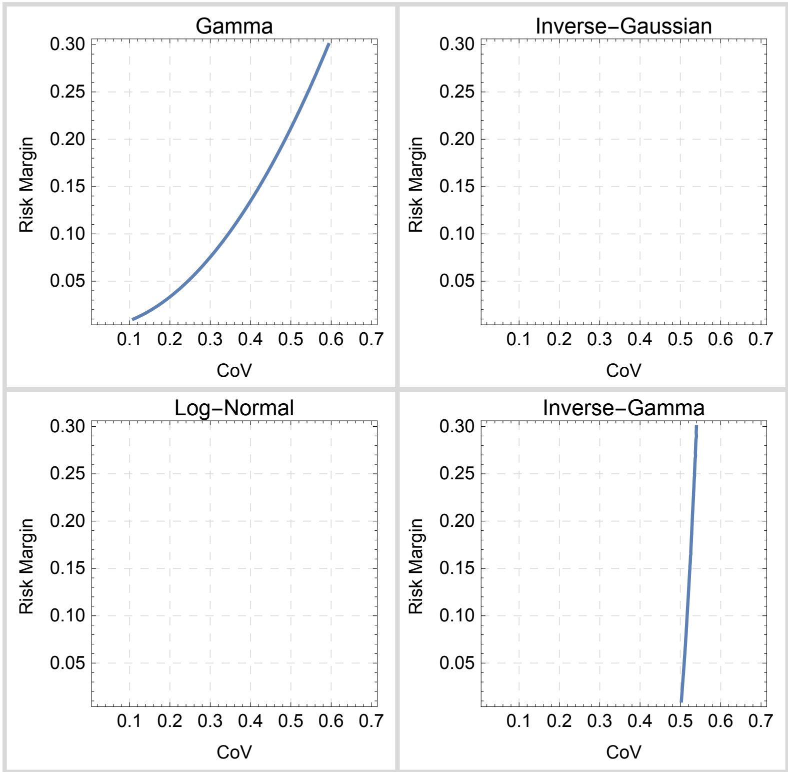
$$\widehat{\alpha} = \Phi(\widehat{z}_\alpha). \tag{30}$$

The functional analysis of the roots of the cubic equation (21) can be done by analysing its discriminant D. As was assumed in section 2 both skewness and kurtosis are functionally related to the coefficient of variation of the random variable X. In particular, these functional relations will take a certain form as the distribution of X is assumed to be adhered to one of the four types of parametric distributions defined in section 2. By taking this into account, we conclude that the discriminant D is simply an analytical function of the coefficient of variation CoV_X and the relative risk margin η_X. The function $D\left(CoV_X, \eta_X\right)$ is analysed on the range of feasible/practical values of CoV_X and η_X:

$$\mathcal{R} = \left\{ \left(CoV_X, \eta_X\right) \mid 0\% \leq CoV_X \leq 70\% \bigwedge 0\% \leq \eta_X \leq 30\% \right\}.$$

The Figure 3 below depicts the contour graph of D surface for each of the four parametric distributions from $\mathcal{PD}$. The discriminant D changes its sign from positive to negative at the contour graph when moving from left to right. This is the case for Gamma and Inverse-Gamma distributions, although for most practical situations - $CoV < 20\%$ and $\eta > 5\%$ for Gamma, and $CoV < 50\%$ for Inverse-Gamma, the discriminant D has a positive value. The contour graph for Inverse-Gaussian and Log-Normal distributions is empty, indicating that D on $\mathcal{R}$ is positive and hence there exists only one real root of cubic approximation.

Figure 3: *Contour graph of the discriminant D.*



3.3 *Fourth-order Cornish-Fisher approximation*

The fourth-order Cornish-Fisher approximation of $\mathrm{VaR}_\alpha\left(\widetilde{X}\right)$ utilises both skewness γ_X and kurtosis ι_X of X and in addition to the terms of the second-order Normal Power approximation (20) involves terms containing z_α^4:

$$\begin{aligned}\mathrm{VaR}_\alpha(\widetilde{X}) \quad \approx \quad & z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6} + \iota_X \frac{z_\alpha^3 - 3z_\alpha}{24} \\ & -\gamma_X^2 \frac{2z_\alpha^3 - 5z_\alpha}{36} - \gamma_X \iota_X \frac{z_\alpha^4 - 5z_\alpha^2 + 2}{24} \\ & +\gamma_X^3 \frac{12z_\alpha^4 - 53z_\alpha^2 + 17}{324}. \end{aligned} \tag{31}$$

To find unknown z_α we solve the following quartic equation

$$\begin{aligned} q \quad = \quad & z_\alpha + \gamma_X \frac{z_\alpha^2 - 1}{6} + \iota_X \frac{z_\alpha^3 - 3z_\alpha}{24} \\ & -\gamma_X^2 \frac{2z_\alpha^3 - 5z_\alpha}{36} - \gamma_X \iota_X \frac{z_\alpha^4 - 5z_\alpha^2 + 2}{24} \\ & +\gamma_X^3 \frac{12z_\alpha^4 - 53z_\alpha^2 + 17}{324}, \end{aligned}$$

or equivalently

$$a z_\alpha^4 + b z_\alpha^3 + c z_\alpha^2 + d z_\alpha + e = 0, \tag{32}$$

where

$$\begin{cases} a &=& \frac{1}{27}\gamma_X^3 - \frac{1}{24}\gamma_X\iota_X, \\ b &=& \frac{1}{24}\iota_X - \frac{1}{18}\gamma_X^2, \\ c &=& \frac{1}{6}\gamma_X + \frac{5}{24}\gamma_X\iota_X - \frac{53}{324}\gamma_X^3, \\ d &=& 1 + \frac{5}{36}\gamma_X^2 - \frac{1}{8}\iota_X, \\ e &=& -\frac{1}{6}\gamma_X + \frac{17}{324}\gamma_X^3 - \frac{1}{12}\gamma_X\iota_X - q \end{cases} \tag{33}$$

The roots of the quartic equation (32) can be found using the Ferrari's method (see, e.g., Abramowitz and Stegun [1], or for the detailed description of a unifying approach to solving quartic equations please refer to Shmakov [20]):

$$x = \frac{-p \pm \sqrt{p^2 - 8t}}{4}, \tag{34}$$

where

$$p = \frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - \frac{4c}{a} + 4y}, \tag{35}$$

$$t = y \mp \sqrt{y^2 - \frac{4e}{a}}, \tag{36}$$

where y is the largest root of the cubic resolvent

$$y^3 - \frac{c}{a}y^2 + \left(\frac{bd}{a^2} - \frac{4e}{a}\right)y + \left(\frac{4ce}{a^2} - \frac{b^2e}{a^3} - \frac{d^2}{a^2}\right) = 0 \tag{37}$$

The root $\widehat{z}_\alpha$ is then the second largest of all real roots of the quartic equation (32). The PoS level, α, is then found as follows

$$\widehat{\alpha} = \Phi(\widehat{z}_\alpha). \tag{38}$$

3.4 *Analysis of quality of approximations and practical implementation*

This subsection provides the analysis of the quality of the three types of PoS approximations proposed. Specifically, the analysis was performed using numerical computations and comparing each type of PoS approximation to the theoretical value of PoS derived from a given parametric distributions from $\mathcal{PD}$.

Given the parametric distribution (CDF) $F_{u,v} \in \mathcal{PD}$[10] and the PoS approximation $A \in \mathcal{A}$[11], the following comparison is made for each pair of coefficient of variation CoV and risk margin η from the range $\mathcal{R}' = \{(CoV, \eta) \mid 0\% \le CoV \le 50\% \wedge 0\% \le \eta \le 15\%\} \subset \mathcal{R}$:

$$\widehat{PoS}_A \;\text{ vs. }\; PoS_F = F_{u',v}(1+\eta),$$

where $F_{u',v}$ is assumed to be normalised so that its mean is one and standard deviation is CoV, and thus the scale parameter u' is a function of shape parameter v (as per (2) and the explanations provided on page 7).

The results of the analysis are provided in the following four subsections. There $\Delta\%$ error of PoS approximation $\mathcal{A}$ is defined as

$$\Delta\% = \frac{\widehat{PoS}_A - PoS_F}{\widehat{PoS}_A} \cdot 100\%.$$

10 $\mathcal{PD}$ was defined on page 8.
11 $\mathcal{A}$ was defined on page 13.

3.4.1 *PoS of Gamma distribution: actual vs. approximated*

Table 3: *PoS of Gamma distribution with η = 5%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx** Δ %	**Cubic approx** Δ %	**Quartic approx** Δ %
CoV =							
5%	0.84144	0.84134	0.84144	0.84143	-0.01157	0.00002	-0.00139
10%	0.70025	0.69992	0.70025	0.70034	-0.04625	0.00080	0.01362
15%	0.64719	0.64664	0.64722	0.64772	-0.08621	0.00338	0.08078
20%	0.62263	0.62181	0.62269	0.62405	-0.13175	0.00873	0.22719
25%	0.61021	0.60909	0.61032	0.61312	-0.18453	0.01780	0.47563
30%	0.60402	0.60253	0.60421	0.60912	-0.24616	0.03147	0.84370
35%	0.60143	0.59952	0.60174	0.60951	-0.31806	0.05063	1.34234
40%	0.60112	0.59871	0.60158	0.61299	-0.40141	0.07610	1.97402
45%	0.60232	0.59933	0.60298	0.61877	-0.49713	0.10871	2.73105
50%	0.60460	0.60093	0.60550	0.62633	-0.60588	0.14924	3.59465

Table 4: *PoS of Gamma distribution with η = 10%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx** Δ %	**Cubic approx** Δ %	**Quartic approx** Δ %
CoV =							
5%	0.97462	0.97460	0.97462	0.97461	-0.00251	-0.00005	-0.00076
10%	0.84172	0.84134	0.84172	0.84163	-0.04474	0.00019	-0.01084
15%	0.75656	0.75574	0.75657	0.75663	-0.10792	0.00184	0.00860
20%	0.70899	0.70772	0.70903	0.70964	-0.17897	0.00581	0.09127
25%	0.68089	0.67914	0.68098	0.68265	-0.25672	0.01300	0.25963
30%	0.66357	0.66130	0.66373	0.66709	-0.34193	0.02423	0.53011
35%	0.65276	0.64991	0.65302	0.65871	-0.43571	0.04033	0.91290
40%	0.64612	0.64264	0.64652	0.65524	-0.53909	0.06207	1.41080
45%	0.64233	0.63814	0.64291	0.65529	-0.65295	0.09024	2.01815
50%	0.64055	0.63557	0.64136	0.65798	-0.77794	0.12557	2.72020

Table 5: *PoS of Gamma distribution with η = 15%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx** Δ %	**Cubic approx** Δ %	**Quartic approx** Δ %
CoV =							
5%	0.99800	0.99800	0.99800	0.99800	0.00013	-0.00002	0.00032
10%	0.92839	0.92814	0.92839	0.92826	-0.02672	-0.00020	-0.01411
15%	0.84217	0.84134	0.84217	0.84187	-0.09740	0.00059	-0.03559
20%	0.78272	0.78127	0.78275	0.78259	-0.18580	0.00333	-0.01599
25%	0.74372	0.74161	0.74378	0.74426	-0.28357	0.00883	0.07313
30%	0.71769	0.71490	0.71782	0.71949	-0.38902	0.01789	0.24957
35%	0.70002	0.69650	0.70024	0.70369	-0.50226	0.03125	0.52396
40%	0.68793	0.68363	0.68827	0.69412	-0.62389	0.04968	0.90002
45%	0.67971	0.67458	0.68021	0.68905	-0.75457	0.07389	1.37402
50%	0.67429	0.66826	0.67500	0.68734	-0.89486	0.10459	1.93461

3.4.2 *PoS of Inverse-Gaussian (I-Gauss) distribution: actual vs. approximated*

Table 6: *PoS of I-Gauss distribution with η = 5%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.84163	0.84134	0.84163	0.84158	-0.03417	0.00006	-0.00595
10%	0.70487	0.70390	0.70494	0.70526	-0.13698	0.01048	0.05546
15%	0.65574	0.65415	0.65605	0.65787	-0.24244	0.04733	0.32493
20%	0.63471	0.63252	0.63550	0.64039	-0.34511	0.12383	0.89386
25%	0.62556	0.62277	0.62713	0.63690	-0.44486	0.25138	1.81310
30%	0.62239	0.61902	0.62513	0.64155	-0.54229	0.43962	3.07814
35%	0.62261	0.61864	0.62695	0.65136	-0.63820	0.69646	4.61713
40%	0.62487	0.62029	0.63130	0.66422	-0.73347	1.02802	6.29689
45%	0.62842	0.62322	0.63747	0.67841	-0.82896	1.43861	7.95459
50%	0.63281	0.62695	0.64503	0.69256	-0.92548	1.93062	9.44294

Table 7: *PoS of I-Gauss distribution with η = 10%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.97335	0.97330	0.97335	0.97332	-0.00498	-0.00056	-0.00343
10%	0.84244	0.84134	0.84244	0.84206	-0.12973	0.00080	-0.04530
15%	0.76178	0.75942	0.76195	0.76199	-0.30939	0.02220	0.02726
20%	0.71845	0.71490	0.71901	0.72082	-0.49335	0.07796	0.33010
25%	0.69417	0.68950	0.69541	0.70052	-0.67204	0.17946	0.91579
30%	0.68032	0.67458	0.68261	0.69248	-0.84333	0.33608	1.78796
35%	0.67266	0.66588	0.67639	0.69216	-1.00703	0.55542	2.89890
40%	0.66889	0.66110	0.67453	0.69668	-1.16369	0.84330	4.15539
45%	0.66769	0.65891	0.67572	0.70400	-1.31412	1.20375	5.43930
50%	0.66823	0.65848	0.67918	0.71258	-1.45925	1.63901	6.63693

Table 8: *PoS of I-Gauss distribution with η = 15%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.99762	0.99764	0.99762	0.99764	0.00153	-0.00017	0.00133
10%	0.92652	0.92584	0.92649	0.92597	-0.07288	-0.00363	-0.05903
15%	0.84368	0.84134	0.84371	0.84245	-0.27672	0.00365	-0.14562
20%	0.78870	0.78463	0.78901	0.78804	-0.51680	0.03932	-0.08361
25%	0.75400	0.74827	0.75487	0.75555	-0.75987	0.11561	0.20618
30%	0.73190	0.72462	0.73367	0.73734	-0.99573	0.24202	0.74276
35%	0.71779	0.70902	0.72085	0.72854	-1.22106	0.42604	1.49854
40%	0.70891	0.69873	0.71368	0.72598	-1.43507	0.67334	2.40792
45%	0.70359	0.69206	0.71054	0.72738	-1.63800	0.98781	3.38250
50%	0.70075	0.68793	0.71037	0.73111	-1.83059	1.37159	4.33173

3.4.3 *PoS of Log-Normal distribution: actual vs. approximated*

Table 9: *PoS of Log-Normal distribution with η = 5%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.84168	0.84134	0.84168	0.84162	-0.04014	-0.00009	-0.00698
10%	0.70505	0.70394	0.70520	0.70554	-0.15804	0.02019	0.06927
15%	0.65604	0.65431	0.65666	0.65871	-0.26287	0.09528	0.40798
20%	0.63510	0.63294	0.63673	0.64236	-0.34001	0.25657	1.14409
25%	0.62600	0.62360	0.62936	0.64088	-0.38288	0.53670	2.37735
30%	0.62285	0.62044	0.62890	0.64862	-0.38719	0.97168	4.13777
35%	0.62305	0.62086	0.63304	0.66254	-0.35047	1.60303	6.33890
40%	0.62524	0.62354	0.64075	0.68001	-0.27184	2.48027	8.76004
45%	0.62867	0.62771	0.65170	0.69847	-0.15194	3.66308	11.10351
50%	0.63287	0.63292	0.66592	0.71582	0.00731	5.22190	13.10631

Table 10: *PoS of Log-Normal distribution with η = 10%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.97334	0.97330	0.97333	0.97330	-0.00455	-0.00099	-0.00412
10%	0.84264	0.84134	0.84264	0.84219	-0.15331	0.00079	-0.05275
15%	0.76224	0.75950	0.76259	0.76258	-0.35945	0.04542	0.04405
20%	0.71914	0.71518	0.72035	0.72232	-0.55074	0.16833	0.44209
25%	0.69505	0.69012	0.69784	0.70360	-0.70877	0.40223	1.23119
30%	0.68134	0.67571	0.68667	0.69797	-0.82595	0.78161	2.44139
35%	0.67377	0.66772	0.68283	0.70085	-0.89846	1.34492	4.01961
40%	0.67004	0.66384	0.68435	0.70903	-0.92471	2.13642	5.81980
45%	0.66882	0.66277	0.69027	0.71988	-0.90484	3.20757	7.63451
50%	0.66929	0.66367	0.70019	0.73128	-0.84037	4.61660	9.26212

Table 11: *PoS of Log-Normal distribution with η = 15%: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.99761	0.99764	0.99761	0.99763	0.00247	-0.00019	0.00157
10%	0.92661	0.92582	0.92654	0.92595	-0.08471	-0.00755	-0.07046
15%	0.84412	0.84134	0.84418	0.84269	-0.32879	0.00700	-0.16957
20%	0.78951	0.78476	0.79021	0.78892	-0.60285	0.08802	-0.07573
25%	0.75514	0.74866	0.75718	0.75759	-0.85764	0.27076	0.32484
30%	0.73332	0.72543	0.73764	0.74120	-1.07512	0.58917	1.07495
35%	0.71942	0.71044	0.72718	0.73488	-1.24739	1.07935	2.14886
40%	0.71069	0.70095	0.72335	0.73521	-1.37096	1.78134	3.44982
45%	0.70546	0.69527	0.72479	0.73953	-1.44491	2.74003	4.82924
50%	0.70266	0.69233	0.73079	0.74573	-1.47009	4.00430	6.12948

3.4.4 *PoS of Inverse-Gamma (I-Gamma) distribution: actual vs. approximated*

Table 12: *PoS of I-Gamma distribution with* η = 5%*: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.84201	0.84134	0.84201	0.84186	-0.07943	-0.00016	-0.01818
10%	0.70999	0.70787	0.71048	0.71128	-0.29833	0.06906	0.18148
15%	0.66496	0.66195	0.66716	0.67206	-0.45311	0.33040	1.06769
20%	0.64750	0.64427	0.65343	0.66675	-0.49908	0.91653	2.97346
25%	0.64146	0.63883	0.65435	0.67984	-0.40973	2.01038	5.98329
30%	0.64098	0.63992	0.66599	0.70288	-0.16496	3.90258	9.65817
35%	0.64343	0.64504	0.68908	0.72776	0.25038	7.09393	13.10543
40%	0.64748	0.65297	0.72778	0.74837	0.84693	12.40180	15.58108
45%	0.65237	0.66302	0.78616	0.76240	1.63101	20.50748	16.86592
50%	0.65766	0.67479	0.85339	0.77020	2.60462	29.76241	17.11199

Table 13: *PoS of I-Gamma distribution with* η = 10%*: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.97211	0.97202	0.97208	0.97199	-0.00894	-0.00316	-0.01150
10%	0.84387	0.84134	0.84392	0.84273	-0.29948	0.00596	-0.13519
15%	0.76828	0.76315	0.76958	0.76910	-0.66883	0.16866	0.10589
20%	0.72955	0.72256	0.73412	0.73736	-0.95826	0.62619	1.07058
25%	0.70925	0.70131	0.72024	0.72957	-1.11964	1.54823	2.86444
30%	0.69882	0.69093	0.72107	0.73559	-1.12789	3.18495	5.26202
35%	0.69402	0.68731	0.73520	0.74760	-0.96663	5.93428	7.72048
40%	0.69259	0.68827	0.76448	0.75964	-0.62481	10.37882	9.67967
45%	0.69324	0.69257	0.81052	0.76848	-0.09563	16.91873	10.85441
50%	0.69515	0.69948	0.86485	0.77331	0.62387	24.41323	11.24424

Table 14: *PoS of I-Gamma distribution with* η = 15%*: actual vs. approximated.*

	Parametric distr	**Quadr approx**	**Cubic approx**	**Quartic approx**	**Quadr approx Δ %**	**Cubic approx Δ %**	**Quartic approx Δ %**
CoV =							
5%	0.99719	0.99725	0.99718	0.99723	0.00568	-0.00062	0.00395
10%	0.92519	0.92360	0.92498	0.92348	-0.17188	-0.02221	-0.18407
15%	0.84670	0.84134	0.84705	0.84304	-0.63269	0.04157	-0.43220
20%	0.79693	0.78814	0.79977	0.79512	-1.10264	0.35653	-0.22747
25%	0.76700	0.75569	0.77535	0.77179	-1.47459	1.08752	0.62421
30%	0.74908	0.73631	0.76745	0.76436	-1.70476	2.45298	2.04028
35%	0.73851	0.72543	0.77377	0.76574	-1.77030	4.77466	3.68807
40%	0.73258	0.72044	0.79474	0.77029	-1.65763	8.48522	5.14823
45%	0.72965	0.71974	0.83068	0.77438	-1.35871	13.84668	6.12983
50%	0.72868	0.72235	0.87483	0.77637	-0.86957	20.05614	6.54504

3.4.5 *Summary of analysis results and practical implementation*

SUMMARY OF RESULTS. The results of the analysis of quality of PoS approximations show that in general across all range of parametric distributions from $\mathcal{PD}$ the second-order Cornish-Fisher (first-order Normal Power) approximation

- is as good as higher-order approximations from $\mathcal{A}$ for the pairs of (CoV, η) that result in PoS of higher than 80%; and
- overperforms higher-order approximations for the pairs of (CoV, η) that result in PoS of below 70%.

This can be explained simply by the fact that the moments like kurtosis and other higher-order moments are likely to drive the higher quantiles in the tail and have little impact on the lower quantiles near the central estimate of the distribution. It appears that skewness does explain the lower quantiles well and higher-order moments become irrelevant. Also, higher PoS is achieved when risk margin η is disproportionally higher than (e.g. a multiple of) the level of distribution variability CoV, and given the practical range of η between 5% and 30%, the corresponding CoV achieving high PoS would be no greater than 15%, which means the resulting skewness would be likely less than 1 for all $F \in \mathcal{PD}$. It is known fact that the Normal Power approximation is generally of good quality when skewness is contained below 1 (see, e.g., Daykin et al. [5]).

However, in principle kurtosis may also well be important in explaining the lower quantiles in the range between 70-th and 80-th. This is likely to be the case, although rather extreme, when

- the reserve distribution profile is overly skewed and leptokurtotic, like Inverse-Gamma distribution; and also
- the $CoV > 50\%$ and the risk margin η is at least half of CoV.

This has been evidenced during the analysis (not shown above), although the approximation error for the Normal Power remains modest and below 3% mark.

NOTES ON IMPLEMENTATION. It is the aim of the paper to provide quick and practical approximations of PoS of risk margin. The implementation of PoS approximations may well be done in a 'standard formula' style. Below outlines the steps of the proposed implementation.

1. The table of adjustment coefficients is pre-computed for each pair of coefficient of variation CoV and risk margin η separately for each approximation $A \in \mathcal{A}$ and parametric distribution $F \in \mathcal{PD}$. The adjustment coefficients are constructed from $\Delta\%$ error by taking $100\% - \Delta\%$. They indicate the factor by which the particular approximation A needs to be multiplied to arrive at the true theoretical PoS level for a given parametric distribution F at (CoV, η).
2. For a given reserve X, for which its risk profile F_X is characterised by its central estimate CE_X, coefficient of variation CoV_X, skewness γ_X and kurtosis ι_X, compute SC_X and $KCsq_X$ ratios to identify the relative location of the reserve risk profile F_X with respect to the four parametric distributions from $\mathcal{PD}$. Let F_X be located between two known parametric distributions F_1 and F_2 from $\mathcal{PD}$.
3. For given coefficient of variation CoV_X and risk margin η_X identify the best PoS approximation A_X[12] for the two adjacent parametric distributions F_1 and F_2 from $\mathcal{PD}$, and then interpolate adjustment coefficients of A_X at

12 Given the results of analysis of quality of PoS approximations, there is always at least one type of PoS approximation of fairly good quality for a given pair (CoV_X, η_X).

(CoV_X, η_X) between F_1 and F_2. Here, the interpolation is done in relation to proximity of $SC_X(CoV_X)$ and $KCsq_X(CoV_X)$ ratios to analogous ratios of F_1 and F_2 at CoV_X.

4. Compute the approximated PoS of reserve risk profile F_X at (CoV_X, η_X) and adjust it by the adjustment coefficient obtained in the preceding step using interpolation.

In summary, the implementation involves computing PoS approximation, grid searching on pre-computed adjustment factors and using interpolation. To secure better quality of interpolation one could add more parametric distributions into the set of existing distributions $\mathcal{PD}$ to make them more even spaced. For example, by inspecting Figure 1 and Figure 2 it is evident that there is quite a big gap between the Log-Normal distribution and the Inverse-Gamma distribution. This gap could be filled in by introducing new parametric distributions in the form of a *mixed distribution*:

$$F_{\text{new}}(x) = wF_{\text{LN}}(x) + (1-w)F_{\text{IGamma}}(x), \quad x > 0, \quad w \in (0,1).$$

The set of adjustment coefficients would then be computed for newly introduced (synthetic) parametric distributions.

4 POS APPROXIMATIONS FOR COC RISK MARGIN - PORTFOLIO OF MULTIPLE RESERVING CLASSES

This section focuses on providing approximation formulae for PoS of diversified CoC risk margin of the portfolio of multiple reserving classes. As indicated in section 2 we assume that the centralised and normalised copy of reserve value X_i of the i-th class, $\widetilde{X}_i$, is estimated by the Fleishman polynomial structure of a standard normal random variable. In particular, we consider the following two different cases:

- $X_i \sim P_2(Z_i) = a_i z_i + b_i\left(z_i^2 - 1\right)$ - suitable for estimating skewness of the reserving portfolio risk profile when the risk margin is approximated using skewness only;
- $X_i \sim P_3(Z_i) = a_i z_i + b_i\left(z_i^2 - 1\right) + c_i z_i^3$ - suitable for estimating skewness and kurtosis of the reserving portfolio risk profile when the risk margin is approximated using both skewness and kurtosis.

The coefficients of polynomials P_2 and P_3 are calibrated using the method of moments by matching the second and third moments of $P_2(Z_i)$ and the second, third and fourth moments of $P_3(Z_i)$ to 1 (standard deviation of $\widetilde{X}_i$), γ_i (skewness of X_i) and $\iota_i + 3$ (non-centralised or absolute kurtosis[13] of X_i) respectively.

The coefficients of P_2 can be analytically expressed by solving the following system of equations:

$$\begin{cases} 1 &= a_i^2 + 2b_i^2, \\ \gamma_i &= 6a_i^2 b_i + 8b_i^3 \end{cases} \tag{39}$$

The system (39) is reduced to

$$\begin{cases} a_i &= \sqrt{1-2b_i^2}, \\ \gamma_i &= 6b_i - 4b_i^3 \end{cases} \tag{40}$$

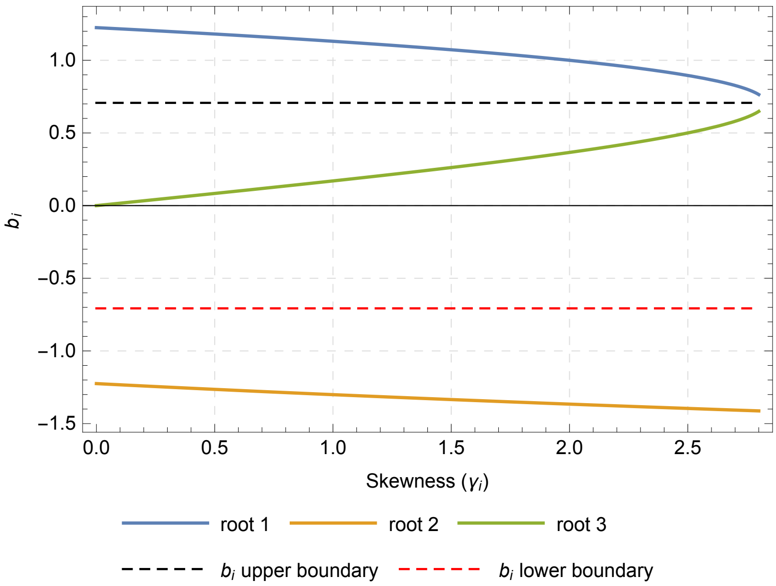
from where we get

$$\begin{cases} a_i &= \sqrt{1-2b_i^2}, \\ b_i &= \text{a real root of cubic equation in (40).} \end{cases} \tag{41}$$

13 As per our definition ι is defined via the fourth- and second-order cumulants and thus represents excess-kurtosis (used in the Cornish-Fisher expansion).

From (40) it follows that $|b_i| \le \frac{1}{\sqrt{2}}$ and thus $0 \le \gamma_i \le 2\sqrt{2}$, which may indicate that it is not suitable for the types of reserve risk profile that are adhering to Inverse-Gamma parametric distribution and have CoV above 50%. The discriminant D of cubic equation in (40) is equal to $\frac{\gamma_i^2}{64} - \frac{1}{8}$ and is negative for $\gamma_i < 2\sqrt{2}$, indicating that there are three real roots as defined in (29): 'root 1' x_1, 'root 2' x_2 and 'root 3' x_3. By analysing those roots as functions of γ_i we could eliminate the roots which fall outside the interval of admissible values of b_i, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Figure 4: *Real roots of cubic equation in (40).*



The Figure 4 above shows that there is only one suitable root for b_i and that is 'root 3'.

The coefficients of P_3 are calibrated numerically. Their values are pre-computed and tabulated in Appendix A for different levels of CoV and the four parametric distributions ('domains of attraction') from $\mathcal{PD}$.

4.1 *Estimation of skewness and kurtosis of reserving portfolio*

We define the total reserve value across the portfolio of m reserving classes as

$$X_\Sigma = \sum_{i=1}^{m} X_i, \tag{42}$$

where each i-th class reserve value is approximated by Fleishman polynomial of a standard normal random variable

$$X_i \approx CE_i \cdot (1 + CoV_i \cdot P_3(Z_i)) \,. \tag{43}$$

It is clear that $CE_\Sigma = \sum_{i=1}^{m} CE_i$. It should be noted that setting $c_i = 0$ reduces the problem to simply approximating $X_i \approx CE_i \cdot (1 + CoV_i \cdot P_2(Z_i))$.

As set up in Assumption 3 in section 2 all the standlone reserve risk profiles interacts between each other according to a Gaussian dependence structure and that the linear correlations ρ_{ij} (coefficients of a Gaussian copula) are derived from the pre-calibrated rank correlation assumptions.

The aim of this subsection is to compute the second, third and fourth central moments of the reserving portfolio distribution and derive their corresponding coefficient of variation CoV_Σ, skewness γ_Σ and kurtosis ι_Σ. Then those values would be further used to compute SC and $KCsq$ ratios to determine the domain of attraction to one of the four parametric distributions from $\mathcal{PD}$ and then estimate the PoS of the diversified risk margin η_Σ at the portfolio level by using the approximation formulae already derived in section 3.

COEFFICIENT OF VARIATION We compute the variance of X_Σ:

$$\begin{aligned}
\mathrm{Var}[X_\Sigma] &= \mathbb{E}\left[\left(\sum_{i=1}^{m}\sigma_i \cdot P_3(Z_i)\right)^2\right] \\
&= \sum_{i=1}^{m}\sigma_i^2 \cdot \mathbb{E}\left[P_3^2(Z_i)\right] + 2\sum_{ij}\sigma_i\sigma_j \cdot \mathbb{E}\left[P_3(Z_i)P_3(Z_j)\right] \\
&= \sum_{i=1}^{m}\sigma_i^2 + 2\sum_{ij}\sigma_i\sigma_j \cdot \mathbb{E}\left[P_3(Z_i)P_3(Z_j)\right], \qquad (44)
\end{aligned}$$

where $\sigma_i = CE_i \cdot CoV_i$, and $\mathbb{E}\left[P_3^2(Z_i)\right] = 1$ as the Fleishman polynomial coefficients are calibrated so that the polynomial has unit variance. The following components of the formula (44) above are derived in Appendix B and provided here:

$$\begin{aligned}
\mathbb{E}\left[P_3(Z_i)P_3(Z_j)\right] &= \rho_{ij}\left(a_i a_j + 2b_i b_j \rho_{ij} + 3\left(a_i c_j + a_j c_i\right)\right. \\
&\quad \left. +3c_i c_j\left(3 + 2\rho_{ij}^2\right)\right). \qquad (45)
\end{aligned}$$

When P_2 polynomial is used, i.e. $c_i = 0$, then this component is reduced to

$$\mathbb{E}\left[P_3(Z_i)P_3(Z_j)\right] = \rho_{ij}\left(a_i a_j + 2b_i b_j \rho_{ij}\right). \qquad (46)$$

Then CoV_Σ^2 is then equal to

$$\begin{aligned}
CoV_\Sigma^2 &= \frac{\mathrm{Var}[X_\Sigma]}{CE_\Sigma^2} \\
&= \sum_{i=1}^{m} w_i^2\, CoV_i^2 \\
&\quad +2\sum_{ij} w_i\, w_j\, CoV_i\, CoV_j \cdot \mathbb{E}\left[P_3(Z_i)P_3(Z_j)\right], \qquad (47)
\end{aligned}$$

where $w_i = \frac{CE_i}{CE_\Sigma}$.

SKEWNESS We compute the third central moment of X_Σ:

$$\begin{aligned}
\mathbb{E}\left[(X_\Sigma - CE_\Sigma)^3\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{m}\sigma_i \cdot P_3(Z_i)\right)^3\right] \\
&= \sum_{i=1}^{m}\sigma_i^3 \cdot \gamma_i \qquad (48) \\
&\quad +3\sum_{ij}\sigma_i^2\sigma_j \cdot \mathbb{E}\left[P_3(Z_i)^2 P_3(Z_j)\right] \\
&\quad +6\sum_{ijk}\sigma_i\sigma_j\sigma_k \cdot \mathbb{E}\left[P_3(Z_i)P_3(Z_j)P_3(Z_k)\right],
\end{aligned}$$

where $\mathbb{E}\left[P_3^3(Z_i)\right] = \gamma_i$ as the Fleishman polynomial coefficients are calibrated so that the polynomial has skewness γ_i for i-th standalone risk profile. In formula (48) above the summation term with multiple 3 has $\binom{m}{2}$ different sub-terms, and the summation term with multiple 6 is relevant if $m \geq 3$ and has $\binom{m}{3}$ different sub-terms.

The following components of the formula (48) above are derived in Appendix B and provided here only for the partial case when P_2 is used, i.e. $c_i = 0$:

$$\begin{aligned}
\mathbb{E}\left[P_3(Z_i)^2 P_3(Z_j)\right] &= 2\rho_{ij}\left(2a_i a_j b_i + \left(a_i^2 + 4b_i^2\right) b_j \rho_{ij}\right) & (49)\\
\mathbb{E}\left[P_3(Z_i) P_3(Z_j) P_3(Z_k)\right] &= 2\left(a_j a_k b_i \rho_{ij}\rho_{ik} + a_j a_i b_k \rho_{jk}\rho_{ik}\right.\\
&\quad \left. + a_i a_k b_j \rho_{ij}\rho_{ik}\right) + 8 b_i b_j b_k \rho_{ij}\rho_{ik}\rho_{jk}. & (50)
\end{aligned}$$

The skewness γ_Σ is then calculated as follows

$$\gamma_\Sigma = \frac{\mathbb{E}\left[(X_\Sigma - CE_\Sigma)^3\right]}{(CE_\Sigma\, CoV_\Sigma)^3}. \tag{51}$$

KURTOSIS To compute the fourth central moment of X_Σ we use the Multinomial formula and apply the expectation operator:

$$\begin{aligned}
\mathbb{E}\left[(X_\Sigma - CE_\Sigma)^4\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{m} \sigma_i \cdot P_3(Z_i)\right)^4\right] & (52)\\
&= \sum_{n_1+n_2+\ldots+n_m=4} \binom{4}{n_1,\, n_2,\, \ldots,\, n_m} \mathbb{E}\left[\prod_{k=1}^{m} \sigma_k^{n_k} \cdot P(Z_k)^{n_k}\right]\\
&= \sum_{i=1}^{m} \sigma_i^4 \cdot (\iota_i + 3)\\
&\quad + 4\sum_{ij} \sigma_i^3 \sigma_j \cdot \mathbb{E}\left[P_3(Z_i)^3 P_3(Z_j)\right]\\
&\quad + 6\sum_{ij} \sigma_i^2 \sigma_j^2 \cdot \mathbb{E}\left[P_3(Z_i)^2 P_3(Z_j)^2\right]\\
&\quad + 12\sum_{ijk} \sigma_i^2 \sigma_j \sigma_k \cdot \mathbb{E}\left[P_3(Z_i)^2 P_3(Z_j) P_3(Z_k)\right]\\
&\quad + 24\sum_{ijkl} \sigma_i \sigma_j \sigma_k \sigma_l \cdot \mathbb{E}\left[P_3(Z_i) P_3(Z_j) P_3(Z_k) P_3(Z_l)\right]
\end{aligned}$$

where $\mathbb{E}\left[P_3^4(Z_i)\right] = \iota_i + 3$ as the Fleishman polynomial coefficients are calibrated so that the polynomial has absolute[14] kurtosis value of $\iota_i + 3$ for i-th standalone risk profile. The components of the portfolio skewness formula (52) are derived in Appendix B. Their formulaic expressions are too long to be presented here.

The kurtosis ι_Σ is then calculated as follows

$$\iota_\Sigma = \frac{\mathbb{E}\left[(X_\Sigma - CE_\Sigma)^4\right]}{(CE_\Sigma\, CoV_\Sigma)^4} - 3. \tag{53}$$

5 CONCLUSIONS

The upcoming IFRS 4 Phase II brings one additional specific requirement to disclose confidence level of reserve risk margins calculated under the Solvency II Cost of Capital approach. This is no doubt an important requirement, and to be compliant with it in the future insurers would need to start making the necessary preparations by estimating the effort needed to implement and accommodate this new upcoming regulatory requirement. However, at the moment this is somewhat a remote issue for majority of insurers as they are currently busy making their final preparations for Solvency II before it comes into play in early 2016, and also because the IFRS Phase II will commence only in early 2017.

14 As per our definition ι is defined via the fourth- and second-order cumulants and thus represents excess-kurtosis (used in the Cornish-Fisher expansion). The fourth centralised moment of $P_2(Z_i)$ is simply the absolute kurtosis, hence ι is translated by 3.

In this paper we take the first step in the actuarial research space and look out for practical ways of implementing this new requirement. In particular, we propose a distribution-free approach to estimating IFRS confidence level of CoC reserve risk margins in a 'standard formula' style. Here, the risk margin confidence level, measured by Probability of Sufficiency (PoS), is estimated by using only the key characteristics of the reserve risk profile: *1) the level of variability measured by CoV; 2) skewness;* and if necessary *3) kurtosis*. The PoS approximation formulae are derived for both standalone reserving class risk margin and the diversified risk margin at the portfolio level. The quality of obtained approximations is analysed and it was shown that in most practical situations the approximations utilising CoV and skewness only are of fairly good quality, and that only in extreme situations when the reserve risk profile is overly skewed and leptokurtotic the kurtosis becomes also a significant driver of the quality of approximation.

The CoV, skewness and kurtosis are invariant with respect to reserve risk profile location, i.e. *the reserve central estimate*, and hence are universal in categorising reserve risk profiles. By relating skewness and kurtosis to CoV level we could trace the 'statistical DNA' of reserve risk profiles to be able to locate them in the system of the four known parametric distributions that in principle cover a wide range of reserve risks. The four parametric distributions are used to derive adjustment coefficients to be applied to their corresponding distribution-free approximation of PoS to arrive at the theoretical value of PoS. Knowing the adjustments coefficients of the known parametric distributions that are in close proximity to the particular reserve risk profile and also the distribution-free approximation of PoS of the reserve risk profile allows us to arrive at the ultimate PoS approximation of high quality.

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Appendix A: Tabulated Fleishman coefficients

The Fleishman coefficients of the polynomial $P_3(Z) = aZ + b\left(Z^2 - 1\right) + cZ^3$ of a standard normal random variable Z are calibrated in such a way that

$$\begin{aligned}
\mathbb{E}\left[P_3^2(Z)\right] &= a^2 + 6ac + 15c^2 = 1; \\
\mathbb{E}\left[P_3^3(Z)\right] &= 2b\left(a^2 + 24ac + 105c^2 + 2\right) = \gamma; \\
\mathbb{E}\left[P_3^4(Z)\right] &= 24\left[ac + b^2\left(1 + a^2 + 28ac\right);\right. \\
&\quad \left. +c^2\left(12 + 48ac + 141b^2 + 225c^2\right)\right] = \iota + 3.
\end{aligned} \tag{54}$$

For a given risk profile with the level of variability CoV we find the corresponding values of $\gamma(CoV)$ and $\iota(CoV)$ and solve the system of equations (54).

The tables below provide calibrated Fleishman coefficients for each of the four parametric distribution from $\mathcal{PD}$ at different levels of CoV.

Table 15: *Fleishman coefficients for the four parametric distributions from $\mathcal{PD}$*

	Gamma			Inverse-Gaussian		
CoV	a	b	c	a	b	c
5%	0.7821269	0.0114362	0.0679523	0.7816586	0.0171455	0.0680652
10%	0.7814394	0.0228591	0.0680955	0.7795694	0.0342187	0.0685466
15%	0.7802942	0.0342555	0.0683341	0.7760992	0.0511486	0.0693467
20%	0.7786927	0.0456123	0.0686680	0.7712656	0.0678665	0.0704621
25%	0.7766366	0.0569165	0.0690971	0.7650925	0.0843071	0.0718882
30%	0.7741280	0.0681552	0.0696211	0.7576098	0.1004096	0.0736193
35%	0.7711696	0.0793157	0.0702400	0.7488526	0.1161185	0.0756483
40%	0.7677644	0.0903856	0.0709533	0.7388605	0.1313843	0.0779670
45%	0.7639159	0.1013527	0.0717609	0.7276770	0.1461641	0.0805664
50%	0.7596281	0.1122051	0.0726623	0.7153487	0.1604223	0.0834364

	Log-Normal			Inverse-Gamma		
CoV	a	b	c	a	b	c
5%	0.7815348	0.0171566	0.0681015	0.7806977	0.0228912	0.0683124
10%	0.7790443	0.0343063	0.0686994	0.7755591	0.0458618	0.0695806
15%	0.7748059	0.0514359	0.0697200	0.7664274	0.0689512	0.0718487
20%	0.7686882	0.0685194	0.0711998	0.7523662	0.0921107	0.0753724
25%	0.7605075	0.0855125	0.0731893	0.7318810	0.1151398	0.0805574
30%	0.7500289	0.1023471	0.0757533	0.7027094	0.1376087	0.0880106
35%	0.7369696	0.1189283	0.0789696	0.6615037	0.1587849	0.0986036
40%	0.7210045	0.1351332	0.0829261	0.6033250	0.1776160	0.1135503
45%	0.7017757	0.1508147	0.0877178	0.5206582	0.1928308	0.1345288
50%	0.6789048	0.1658095	0.0934413	0.4009040	0.2031524	0.1640103

If necessary, the Fleishman coefficients can be pre-computed at much higher resolution of CoV and $\mathcal{PD}$.

Appendix B: Derivation of skewness and kurtosis at the portfolio level

The derivation of approximations of skewness and kurtosis of the aggregate reserve risk is based on the assumption that standalone reserve risk profiles are approximated with the Fleishman polynomial of a standard normal random variable and follow a Gaussian dependence structure. This implies a multivariate normal distribution and calculating moments of multivariate polynomial function of a multivariate normal distribution. In particular, the components of skewness and kurtosis approximation formulae for the reserve risk portfolio are of the following types:

$$\mathbb{E}\left[P_3(Z_i)^{n_i}P_3(Z_j)^{n_j}P_3(Z_k)^{n_k}P_3(Z_l)^{n_l}\right]$$

with n_i, n_j, n_k, $n_l \geq 0$ and $n_i + n_j + n_k + n_l \leq 4$. This reduces to calculating high-order moments of the multivariate standard normal distribution:

$$\mathbb{E}\left[Z_i^{n_i}Z_j^{n_j}Z_k^{n_k}Z_l^{n_l}\right]$$

with n_i, n_j, n_k, $n_l \geq 0$ and $n_i + n_j + n_k + n_l \leq 12$. The multivariate standard normal moments for which $n_i + n_j + n_k + n_l$ is odd are equal to zero. When $n_i+n_j+n_k+n_l$ is even the high-order moment of a multivariate standard normal is then calculated using Isserlis's Formula. The Isserlis' Formula is defined and used to compute high-order moments of multivariate standard normal random variables in Isserlis [10].

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