Translation invariant
and
positive homogeneous risk measures
and
portfolio management

Zinoviy Landsman
Department of Statistics,
University of Haifa

-It is wide-spread opinion the use of any **positive homogeneous and translation invariant** risk measure reduces to the same portfolio management as mean-variance risk measure does, which is equivalent to Markowitz portfolio

-We show that generally speaking it is not true, because these measures reduce to a different optimization problem.

-We provide the explicit closed form solution of this problem

-The results will be demonstrated with the data of stocks from NASDAQ/Computer.

Translation invariant and positive homogeneous risk measures

1) $\rho(X+\alpha)=\rho(X)+\alpha, \quad \alpha - const$

2) $\rho(cX)=c\rho(X), \quad c>0-const$

Examples BASEL II

VaR

$$\rho(X) = VaR_q(X)$$

$$= \inf\{x \mid F_X(x) \geq q\}$$

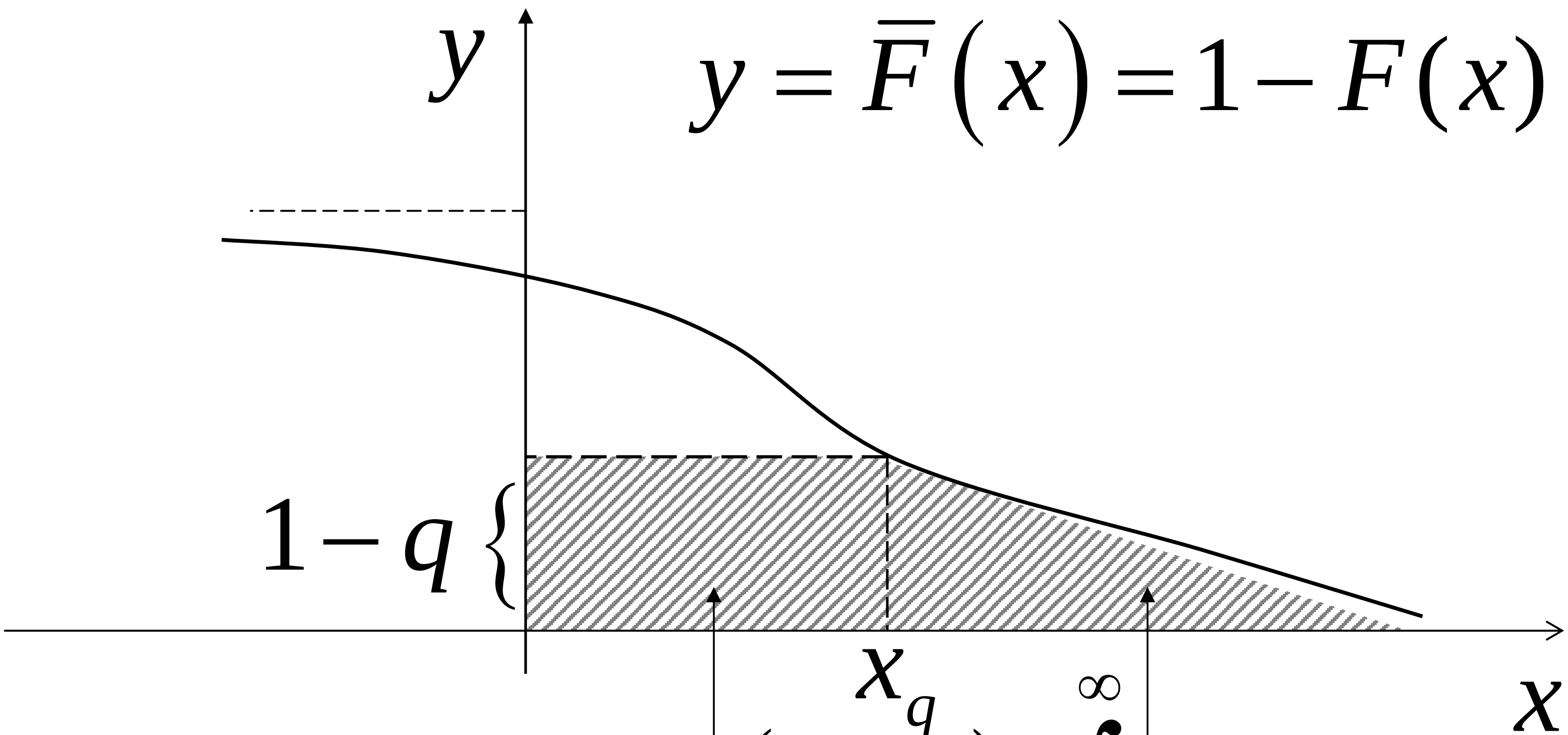
Expected Short Fall

$$\rho(X) = ES_q(X)$$

$$= E(X \mid X > VaR_q(X))$$

Tail VaR, Tail conditional expectation (TCE)

Conditional VaR (CVaR)



$$y = \bar{F}(x) = 1 - F(x)$$

$$VaR_q(X)(1-q) \qquad \int_{x_q}^{\infty} \bar{F}(x)\,dx$$

$$(1-q)\,ES_q(X) = x_q(1-q) + \int_{x_q}^{\infty} \bar{F}(x)\,dx$$

Panjer and Jing (2001) H¨urlimann (2001)

L. and Valdez (2003, 2005)

STD-premium

$$\rho(X) = E(X) + \lambda STD(X)$$

Distorted risk measures

$$\rho(X) = \int_0^{\infty} g(\bar{F}(x))dx,$$

Denneberg (1994) and Wang (1996)

Coherent risk measures

Artzner, Delbean, Eber, and Heath (1999)

Tail SDP Furman and Landsman (2006)

$$\rho(X) = E(X \mid X > VaR_q(X)) + \alpha\sqrt{V(X \mid X > VaR_q(X))}$$

Tail VaR

Tail Variance

Tail Expectation

$$ES_q(X)$$

$$= \arg\inf_c E((X-c)^2 \mid X > VaR_q(X))$$

$$V(X \mid X > VaR_q(X))$$

$$= \inf_c E((X-c)^2 \mid X > VaR_q(X))$$

Portfolio management

$$X = (X_1, \ldots, X_n)^T \sim N_n(\mu, \Sigma)$$

$$P = \sum_{j=1}^{n} x_j X_j, \qquad \sum_{j=1}^{n} x_j = 1$$

$$\rho(P) \to \inf$$

$$\rho(P) = \rho(x^T X) = \rho(x^T X - \mu^T x + \mu^T x)$$

$$= \rho(x^T X - \mu^T x) + \mu^T x$$

$$= \mu^T x + \rho\left(\frac{x^T X - \mu^T x}{\sqrt{VaR(x^T X)}} \sqrt{VaR(x^T X)} \right)$$

$$Z \sim \mathrm{N}(0,1) \quad = \mu^T x + \rho\left(\frac{x^T X - \mu^T x}{\sqrt{VaR(x^T X)}} \right) \sqrt{VaR(x^T X)}$$

$$= \mu^T x + \rho(Z)\sqrt{\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}} \quad \to \inf$$

Translation invariant and positive homogeneous risk measure reduces to minimization

of linear plus **root** of quadratic functional

$$f(\mathbf{x}) = \mu^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \rightarrow \inf$$

$$\mathbf{1}^T \mathbf{x} = 1$$

$$Z \sim \mathrm{N}(0,1) \qquad \lambda = Z_q \Leftrightarrow \inf VaR_q(P).$$

$$\lambda = ES_q(Z) \Leftrightarrow \inf ES_q(P).$$

$\rho(Z)$ – Any translation invariant and positive homogeneous risk measure

Classical mean - variance optimization

$$q(\mathbf{x}) = E(P) + \lambda Var(P) = \mu^T \mathbf{x} + \lambda \mathbf{x}^T \Sigma \mathbf{x} \to \inf$$

linear + quadratic functional

subject to

$$\mathbf{1}^T \mathbf{x} = 1$$

Exact solution (see Boyle et.al (1998))

$$\mathbf{x}^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} + \frac{1}{2\lambda}\left(\frac{\boldsymbol{\mu}^T \Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \Sigma^{-1}\mathbf{1} - \Sigma^{-1}\boldsymbol{\mu} \right)$$

Minimization of linear plus root of quadratic functional

We found the explicit closed form solution for
minimization problem under system of equality constraints (L(2007))

$$\begin{cases} f(\mathbf{x}) = \mu^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \to \inf \\ B\mathbf{x} = \boldsymbol{c} \end{cases}$$

$B = matrix\ m \times n, \quad \boldsymbol{c}$ - vector

The special case for only one constraint m=1

$$\begin{cases} f(\mathbf{x}) = \mu^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \to \inf \\ \mathbf{1}^T \mathbf{x} = 1 \end{cases}$$

$$\Sigma = \begin{pmatrix} \Sigma_{\mathbf{11}} & \sigma \\ \sigma^T & \sigma_{nn} \end{pmatrix},$$

$$\Sigma_{\mathbf{11}} - (n-1) \times (n-1)$$

Define $(n-1)\times(n-1)$ matrix

$$Q=\Sigma_{11}-1_1\sigma^T-\sigma 1_1^T+\sigma_{nn}1_1 1_1^T$$

Lemma $\Sigma>0 \Rightarrow Q>0.$

$$\Delta=(\mu_n-\mu_1,\ldots,\mu_n-\mu_{n-1})^T.$$

Theorem 1. L (2007a). If

$$\lambda>\sqrt{\Delta^T Q^{-1}\Delta},$$

the minimization problem has the finite exact solution

$$\mathbf{x}^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} + \frac{1}{\sqrt{(\lambda^2 - \Delta^T Q^{-1}\Delta)(\mathbf{1}^T\Sigma^{-1}\mathbf{1})}}(\Delta^T Q^{-1}, -\mathbf{1}_1^T Q^{-1}\Delta)^T$$

3. Value-at-Risk and Expected Short Fall

$$\lambda = ES_q(Z) = VaR_q(Z)$$

$$+\frac{1}{1-q}\int_{VaR_q(Z)}^{\infty}(1-F(x))dx \geq VaR_q(Z) = Z_q.$$

Theorem 2

If VaR-optimal portfolio solution is finite, the ES-optimal portfolio solution is automatically finite

Consider a portfolio of 10 stocks from NASDAQ/Computers

Table 1: Portfolio mean return

Stock	ADOBE	Compuware	NVIDIA	Staples	VeriSign
Mean	0.0061	-0.0081	-0.0096	0.0058	0.0064

Stock	Sandisk	Microsoft	Citrix	Intuit	Symantec
Mean	-0.0198	0.0002	-0.0038	-0.0041	0.0061

Table 2: Portfolio covariance return

	ADOBE	Compuware	NVIDIA	Staples	VeriSign
ADOBE	0.006102	0.001173	0.000118	0.000513	0.000121
Compuware	0.001173	0.003310	0.001047	0.000498	0.000847
NVIDIA	0.000118	0.001047	0.002145	0.000122	0.000772
Staples	0.000513	0.000498	0.000122	0.002940	-0.000547
VeriSign	0.000121	0.000847	0.000772	-0.000547	0.003486

	Sandisk	Microsoft	Citrix	Intuit	Symantec
Sandisk	0.004013	-0.000033	0.000844	0.000131	0.000083
Microsoft	-0.000033	0.000485	0.000220	0.000167	0.000062
Citrix	0.000844	0.000220	0.001365	0.000397	0.000445
Intuit	0.000131	0.000167	0.000397	0.000876	0.000027
Symantec	0.000083	0.000062	0.000445	0.000027	0.002542

The loss on the portfolio

$$L = -R = -\sum_{j=1}^{n} x_j X_j,$$

$$VaR_q(L) = -\mu^T \mathbf{x} + z_q \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \xrightarrow[\mathbf{x}^T \mathbf{1}=1]{} \inf$$

$$ES_q(L) = -\mu^T \mathbf{x} + \lambda_q \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \xrightarrow[\mathbf{x}^T \mathbf{1}=1]{} \inf$$

From Theorem 1: Lower bound for solution

$$B = \sqrt{\Delta^T Q^{-1} \Delta} = 0.2142$$

$$X \sim N_n(\mu, \Sigma)$$

VaR optimal portfolio finite $q > 0.585$

ES optimal portfolio finite $q > 0.112$

$q = 0.8$

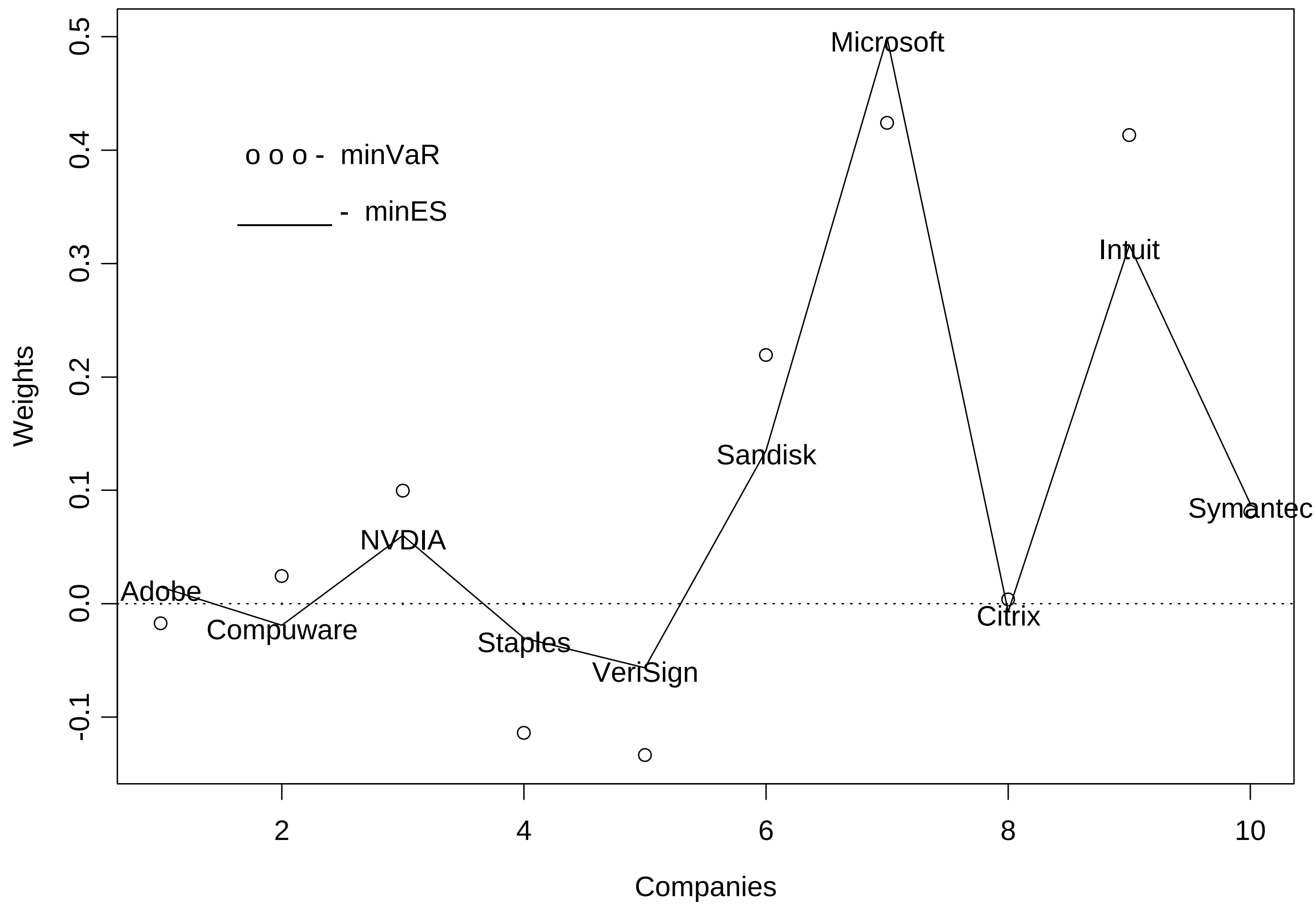
Table 3: Optimal portfolio

	ADOBE	Compuware	NVIDIA	Staples	VeriSign
minVaR	-0.0173	-0.024	0.0996	-0.1141	-0.1337
minTCE	0.0148	-0.0191	0.0601	-0.0305	-0.0567

	Sandisk	Microsoft	Citrix	Intuit	Symantec
minVaR	0.2192	0.4242	-0.0037	0.4133	0.081
minTCE	0.135	0.4993	-0.0071	0.3161	0.088

Table 4: Main characteristics for VaR and ES optimal portfolios

	VaR	TCE	Variance	Mean
minVaR	0.01109	0.0236	0.00052	0.00827
minTCE	0.01162	0.022	0.00035	0.0042



Closeness between VaR and ES optimal portfolios

Distance between portfolios

$$\arg\min_{1^T\mathbf{x}} VaR_q(L),\ \arg\min_{1^T\mathbf{x}} ES_q(L) \in R^n$$

$$d = \sqrt{(\arg\min_{1^T\mathbf{x}} VaR_q(L) - \arg\min_{1^T\mathbf{x}} ES_q(L))^T (\arg\min_{1^T\mathbf{x}} VaR_q(L) - \arg\min_{1^T\mathbf{x}} ES_q(L))}$$

Denote

$$C = (\mathbf{1}^T \Sigma^{-1} \mathbf{1})^{-1/2} \sqrt{\Delta^T Q^{-2} \Delta + (\mathbf{1}_{\mathbf{1}}^T Q^{-1} \Delta)^2}$$

Theorem 3

$$d = [(z_q^2 - B^2)^{-1/2} - (\lambda_q^2 - B^2)^{-1/2}]C \to 0,$$

$$q \to 1$$

Boundary

Elliptical family

$$\mathbf{X} \sim E_n\left(\mu, \Sigma, g_n\right).$$

$$f_{\mathbf{X}}\left(\mathbf{x}\right) = \frac{c_n}{\sqrt{\left|\Sigma\right|}} g_n\left[\frac{1}{2}\left(\mathbf{x}-\mu\right)^T \Sigma^{-1}\left(\mathbf{x}-\mu\right)\right],$$

density generator

$$g_n(u) = \exp(-u) \sim N_n(\mu, \Sigma)$$

Property

$$BX + \mathbf{c} \sim E_m(B\boldsymbol{\mu} + \mathbf{c}, B\Sigma B^T, g_m)$$

guarantees

$$\rho(x^T X) = \boldsymbol{\mu}^T x + \lambda\sqrt{\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}}$$

$$\lambda = \rho(Z), \quad Z \sim E_1(0,1,g_1)$$

$$g_n(u) = \left(1 + u/k_{n,p}\right)^{-p} \sim t_n(\boldsymbol{\mu}, \Sigma, p)$$

Multivariate Generalized t-distribution (GST)

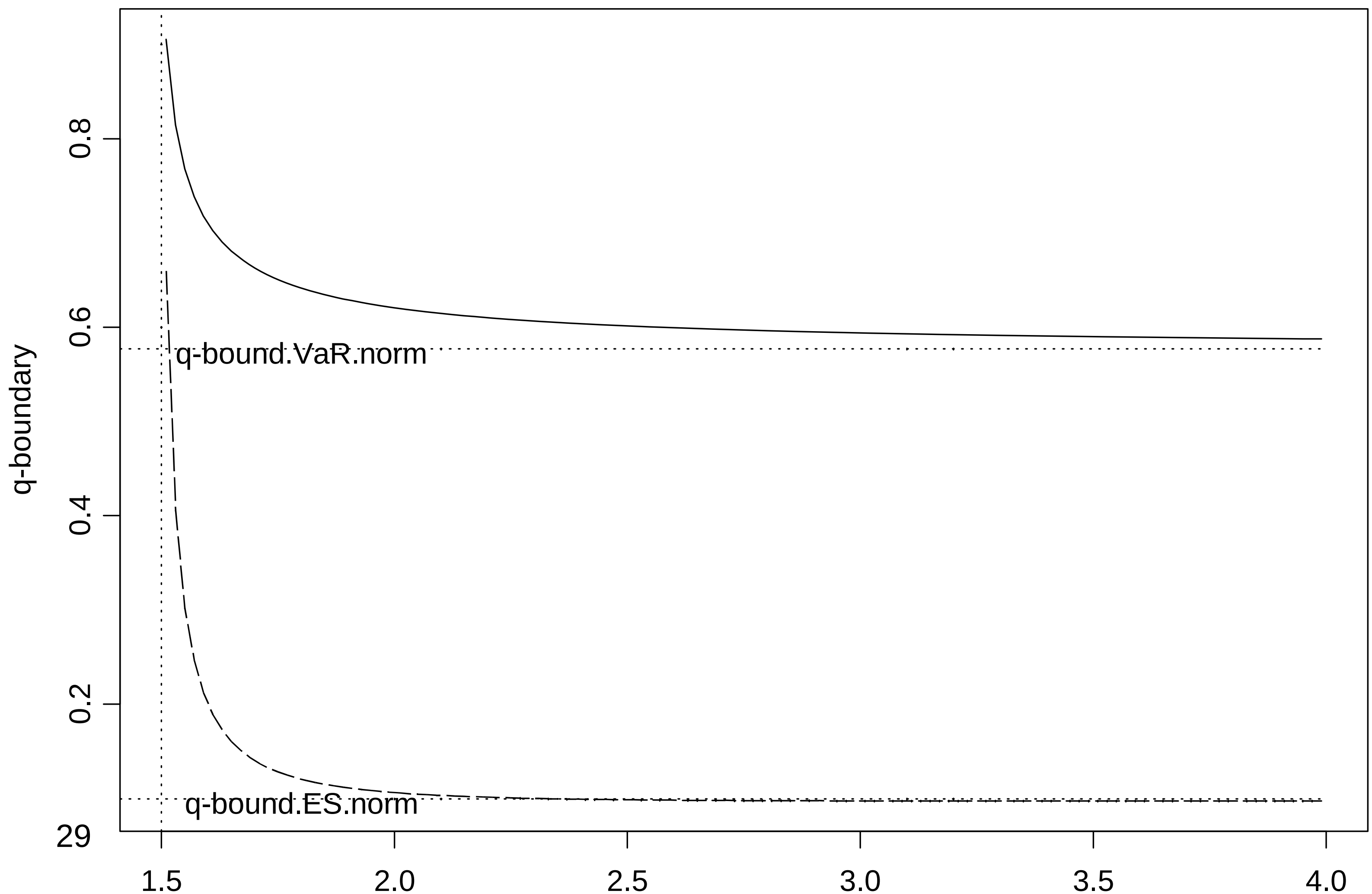
$$\begin{cases} f(\mathbf{x}) = -\mu^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \to \inf \\ \mathbf{1}^T \mathbf{x} = 1 \end{cases}$$

$$T \sim t_1(0, 1, p_1)$$

$$T_q \Rightarrow VaR - optimal$$

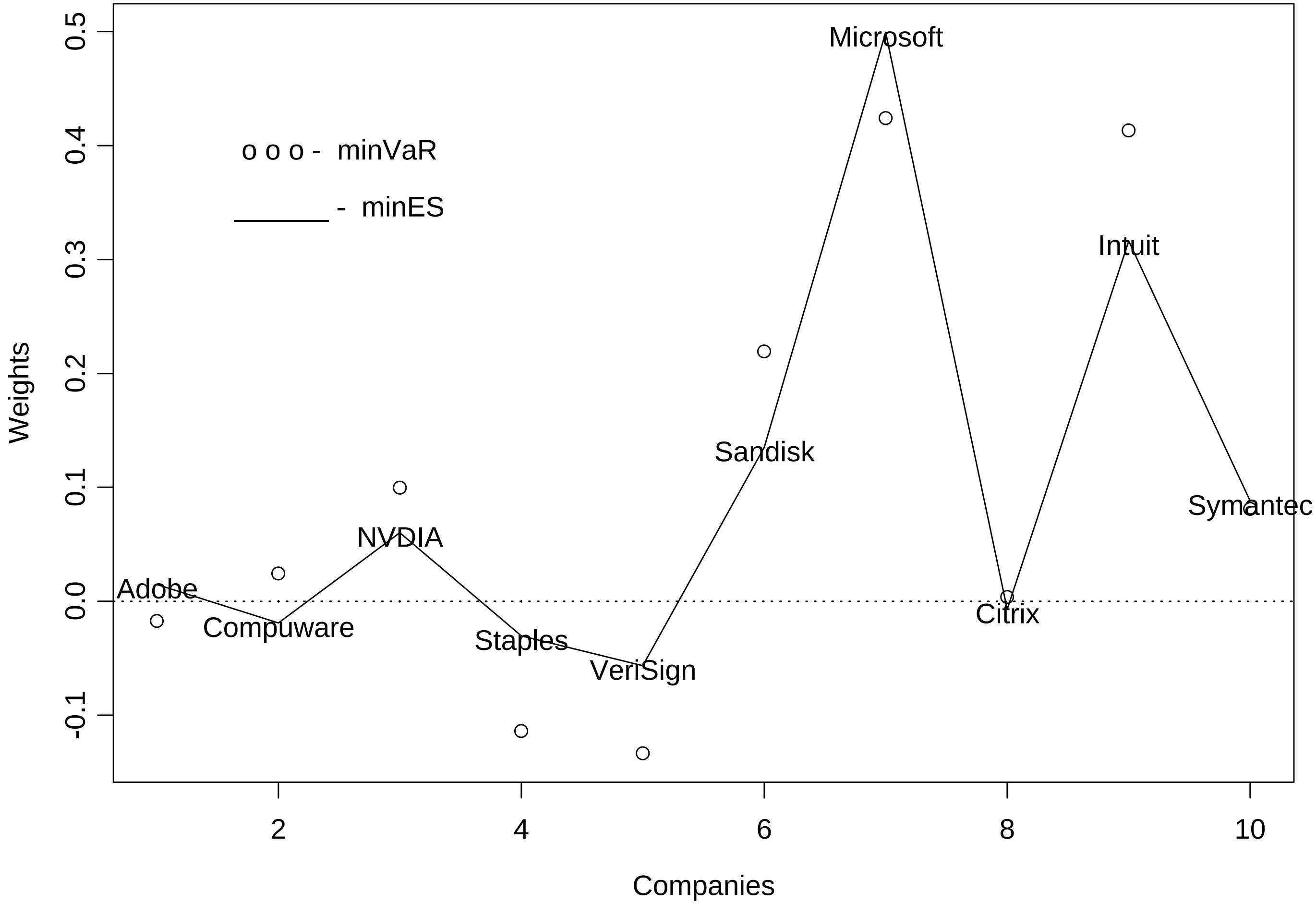
$$ES_q(T) \Rightarrow ES - optimal$$

Figure 1. Lower boundary of VaR and ES q- probability level for GST .



Closeness between VaR and ES optimal portfolios

Figure 2. VaR and ES optimal portfolios for Norm.



Distance between portfolios

$$\arg\min_{1^T\mathbf{x}} VaR_q(L),\ \arg\min_{1^T\mathbf{x}} ES_q(L) \in R^n$$

$$d =$$

$$\sqrt{(\arg\min_{1^T\mathbf{x}} VaR_q(L) - \arg\min_{1^T\mathbf{x}} ES_q(L))^T (\arg\min_{1^T\mathbf{x}} VaR_q(L) - \arg\min_{1^T\mathbf{x}} ES_q(L))}$$

Denote

$$\delta_q = \lambda_q - z_q = ES_q(Z) - VaR_q(Z)$$

$$C = (\mathbf{1}^T \Sigma^{-1} \mathbf{1})^{-1/2} \sqrt{\Delta^T Q^{-2} \Delta + (\mathbf{1}_{\mathbf{1}}^T Q^{-1} \Delta)^2}$$

Theorem 4 $\mathbf{X} \sim E_n\left(\mu, \Sigma, g_n\right).$

$$1)\ \ d = [(z_q^2 - B^2)^{-1/2} - (\lambda_q^2 - B^2)^{-1/2}]C \to 0, q \to 1$$

$$2)\ \ Let\ \ -z_q^2 \frac{d \log g_1(\frac{1}{2} z_q^2)}{dx} \xrightarrow[q \to 1]{} l$$

$$d = \frac{1}{\sqrt{z_q^2 - B^2}}[\frac{1}{l-1} + \frac{1}{2}(\alpha_q - \frac{2l-3}{(l-2)^2}) + O((\alpha - \frac{2l-3}{(l-2)^2})^2)]C,$$

$$\alpha_q = \frac{\delta_q(2z_q + \delta_q)}{z_q^2 - B^2} \to \frac{2l-3}{(l-2)^2}.$$

3) *If* $\mathbf{X} \sim \mathbf{t}_n\left(\mu, \Sigma; p, k_{n,p}\right)$,

$l = 2p_1,\ p_1 = p - (n-1)/2, and$

$$d = \frac{1}{\sqrt{z_q^2 - B^2}}[(\frac{1}{2p_1 - 1} + \frac{1}{2}\frac{(p_1 - 1)^3}{(p_1 - 1/2)^3}(\alpha_q - \frac{p_1 - 3/4}{(p_1 - 1)^2})$$

$$+ O((\alpha_q - \frac{p_1 - 3/4}{(p_1 - 1)^2})^2)]C.$$

4) If $\mathbf{X} \sim \mathbf{N}_n\left(\mu, \Sigma\right), l = \infty,\ and$

$$d = \frac{1}{\sqrt{z_q^2 - B^2}}[\frac{1}{2}\alpha_q + O(\alpha_q^2)]C$$

Figure 3. Distance and it's approximation; GST with power parameter p=1.6

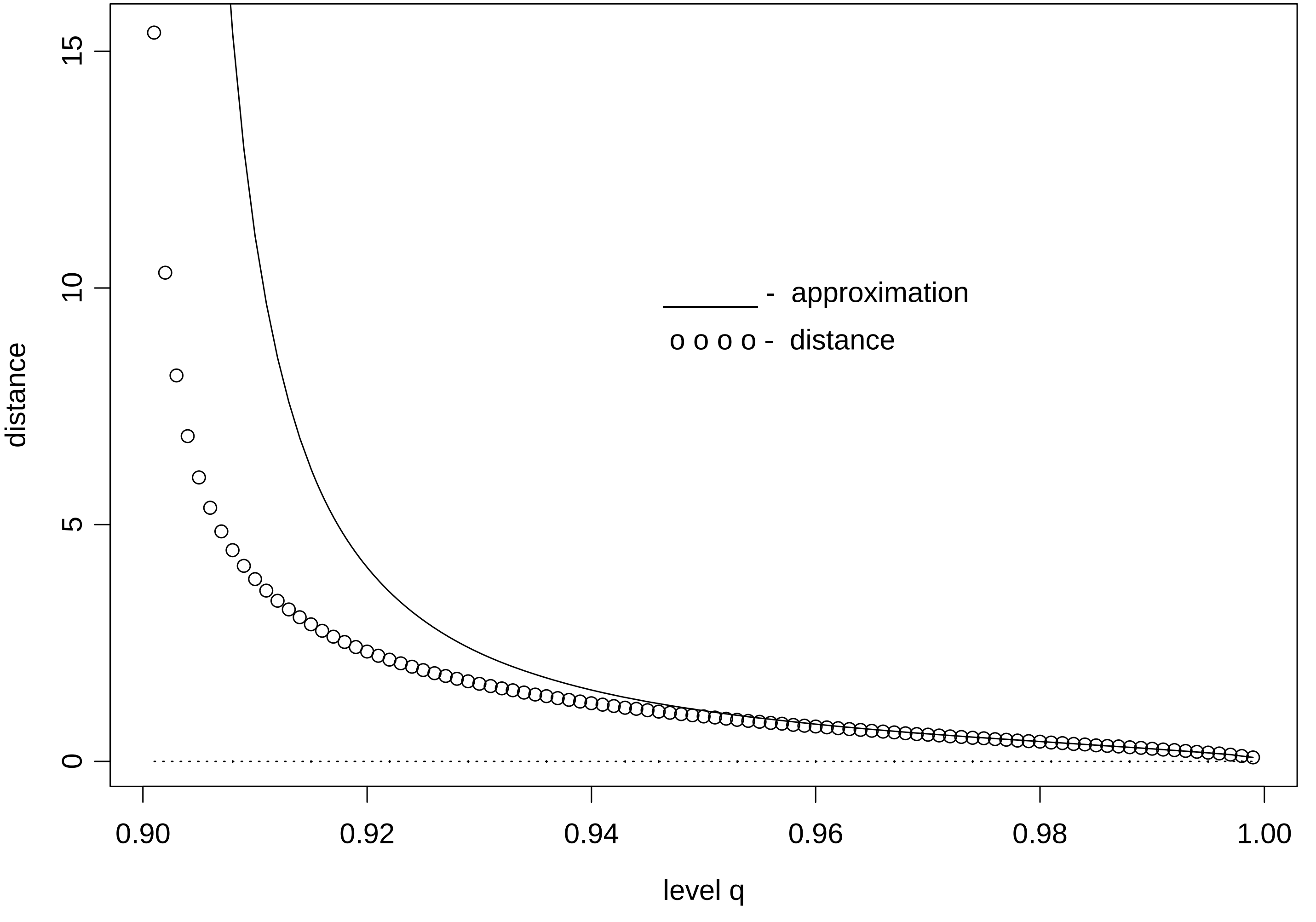
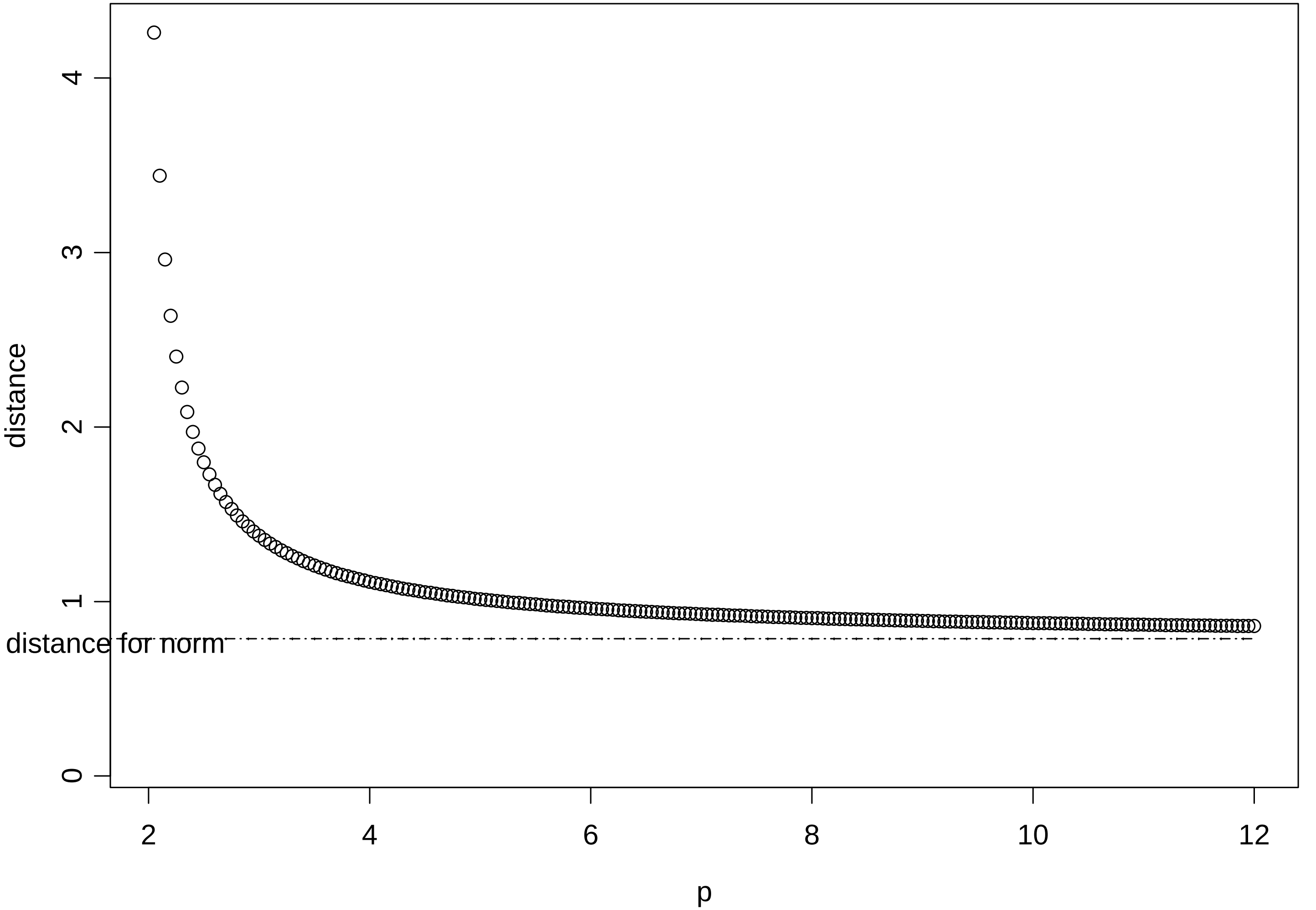


Figure 4. Changing distance with increasing of power parameter p of GST



Conclusion

-Problem of minimization of translation invariant and positive homogeneous risk measures has exact close form solution

-This solution can be easy realized and used for analyzing the influence of parameters of underlying distribution on the portfolio selecting

-The solution is different with that of mean-variance except the case when the portfolio's expected mean is certain

References

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