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Asset Allocation in the light of Liability Cash Flows

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Abstract

Asset allocation is one of the most important investment decisions that financial institutions have to make. Modern portfolio theory suggests that an optimal asset portfolio is one which maximises the return of the portfolio at a certain level of risk which is defined as the variance of the portfolio. In the light of liability cash flows, modern portfolio theory can be extended by regarding a liability as a negative asset. However, non-normal features of both asset return and liability features are always witnessed in reality and the appropriateness of defining risk as the variance of the ultimate surplus that assets have over liability is always questionable.

In this paper, instead of defining risk as the variance of portfolio, the authors define risk as the probability of insolvency and derive the optimal asset portfolios thereafter. Both assets and liabilities are assumed to follow certain stochastic process. Four different stochastic investment models are examined and compared. Asset portfolios based on different approaches are also contrasted.

Introduction

Asset allocation is one of the most important investment decisions that financial institutions have to make. Modern portfolio theory suggests that an optimal asset portfolio is one which maximises the return of the portfolio at a certain level of risk which is defined as the variance of the portfolio. In the light of liability cash flows, both liabilities and assets need to be taken into account. Early and more recent asset/liability portfolio models include Wise(1984a/b), Wilkie (1985), Sharpe and Tint (1990), Elton and Gruber (1992), Leibowitz et al. (1992), Keel and Müller (1995), Hürlimann (2001,2002) etc.

Wise (1984) defines 'closest match asset portfolio" to a liability as the one which will minimise the square of ultimate surplus. The ultimate surplus is measured in terms of the realizable market value of the assets remaining when all liabilities have been extinguished. This paper also illustrates an approach to find a matching portfolio with a worked example. This approach assumes derministic liability cash flows and a stochastic investment return model. This algorithm of the approach is not affected if either the cash flow of assets or liabilities is linked to inflation. The paper also points out the market value of a matching portfolio may be thought of as another technique of valuation. Though it may be argued that a valuation by matching is inappropriate because if the result is a mean ultimate surplus of about zero then there will be roughly a 50% chance of a deficiency, a margin can be applied to the market value of the matching portfolio to ensure positive surplus at a desired level of probability. The paper also illustrates how this approach can be applied to identify a matching portfolio for a pension fund.

Wilkie (1985) points out that the closest matching portfolio identified by Wise(1984) might not be the efficient portfolio, and even if it is efficient under some circumstances, it might not be the most optimal portfolio for a particular investor.

Wilkie (1985) argues that rational investor must take account of the prices of securities in order to choose an optimal portfolio. Therefore, Wilkie considers feasible portfolios in the space P-E-V, where P represents the aggregate price of all assets in the portfolio, E the expected ultimate surplus of assets net of liabilities on completion of the liability cash flows and V the variance of ultimate surplus. Wilkie has therefore generalized conventional portfolio theory by including the price P of the portfolio as a third dimension. In the conventional theory (described by for example by Moore), only E and V are considered because, in the absence of fixed unmarketable liabilities, the proportions of assets to be held in the selected portfolio will be the same whatever the value of P. In order to identify the efficient portfolio, he assumes that investors are in favour of a high expected surplus, E, low variance of surplus, V, and a low immediate price, P. Wilkie also shows how the particular preferences of an investor can be expressed and used to select particular portfolios from the range of efficient portfolios.

Keel and Müller (1995) discuss in detail the set of efficient portfolios in an asset/liability model. The authors put the asset/liability problem in a very close relationship to traditional mean variance portfolio theory. The authors assume the first and second moments of the growth rate of liabilities and all assets are known. The covariance matrix of both the growth rates of liabilities and assets are also assumed to be known. Under the methodology of Markowitz, the authors derive efficient portfolios which minimize the variance of surplus of assets over a liability in the next period for any desired mean of the surplus. They point out that the occurrence of liabilities leads to a parallel shift of the efficient set. They also show how a shortfall constraint such as that the probability of the ratio of asset value to liability is below one can be reconciled with efficiency. They also extend the standard version of CAPM and show how the risk premium for assets whose return is strongly correlated with the growth rate of market representative liabilities might be determined.

Hürlimann (2001) proposes a portfolio selection model based on the expected return of the assets and the economic risk capital (ERC) associated to the asset liability portfolio, in the context of asset and liability management. For short, the author calls it mean-ERC asset liability portfolio selection. Economic risk capital refers to the amount of fund that a portfolio manager has to borrow in order to be able to cover any loss with a high probability. There exist several risk management principles applied to determine ERC.

Two simple methods are the value-at-risk and the expected shortfall approach. The author finds that Mean-ERC efficiency in asset and liability management is closely related.

Hürlimann (2002) defines portfolio shortfall risk as the expected shortfall deviation of a portfolio from the mean return and the natural risk contribution of each portfolio asset to the portfolio shortfall risk as the shortfall risk of the asset. By replacing the variance as a measure of risk in the classical portfolio selection model with the shortfall risk the author proposes an alternative approach to portfolio selection, namely mean-shortfall portfolio selection. The author proves that for the multivariate elliptical return distributions, both the mean-variance and mean-shortfall approach lead to the same conclusions. However, using more general marginal distributions of return, say lognormal returns, the two approaches will yield different results.

Service and Sun (2003) argue that overriding any considerations of theoretical asset/liability profiles insurers must ensure that they remain solvent at all times. As a result if the assets become less than the liabilities the 'game' is over. Hence any definition of "closest match" must take into account the probability of insolvency. They, therefore, define a "closest" asset match as the asset portfolio which, for a given probability of insolvency, requires lowest initial asset value. In their worked example, the authors show an approach to identify the closest matching asset portfolio for a particular portfolio of liabilities when both are comprised of stochastic cashflows.

In this paper, we adopt the definition of "closest" asset match proposed by Service and Sun (2003) and compare the resulting "closest" asset match by using different stochastic asset models. The stochastic asset models examined include Random Walk model ["RW"], Carter's model (1991) ["JC"], Vector AR(1) model ["VAR"], Regime Switching Vector AR(1) model (Harris, 1999) ["RSVAR"].

Data is also split into in-sample data and out-of-sample data, so that the goodness-of-fit of the asset models examined may be compared.

Using a VAR(1) model, we also compare the resulting asset portfolios if we use different three different approaches, namely Service and Sun's "closest asset match" approach, minimization of expected squared ultimate surplus approach, and minimization of expected shortfall approach.

3 Models and Data

Data and methodology applied in this paper is very similar to that applied in Service and Sun (2003), but three more asset models are added for comparison. For the convenience of the readers, we restate it here with some modification due to the addition of more stochastic asset models.

3.1 Liability Data

In this paper, we suppose there exists a hypothetical portfolio of long tail outstanding claims which runs off in ten years' time and where payments of claims are made at the end of each quarter. The hypothetical claims payment experience was set out in Service and Sun (2003).

The claims cashflow model uses the stochastic chain-ladder method suggested by Renshaw and Verrall (1998) and cashflows are then further adjusted by both general inflation and super-imposed inflation. General inflation is simulated stochastically by using the asset model and superimposed inflation is assumed to be 8% p.a. constantly.

3.2 Data required to estimate the parameters of the Asset Models

Economic Variables	**Measurement**
Inflation Rate	CPI
Short-term fixed interest rate	90-Days Bank Accepted Bills
Long-term fixed interest rate	10-year Government Bond
Share dividend yield	Difference between return of AOI Accumulation index and AOI price index
Dividend	Dividend Yield times AOI Price Index in previous year
Share price return	Return of AOI Price Index

Real Gross National Income	Australian Real Gross National Income

The CPI, Real Gross National Income and the AOI Price Index were collected from Datastream and the AOI Accumulation Index was collected from the Australian Stock Exchange. Both the yield of 90-day Bank Accepted Bill and 10-year government bond were obtained from the Reserve Bank.

3.3 Stochastic liability model

Renshaw and Verrall (1998) present a statistical model underlying the chain-ladder technique. They show that the estimates produced by the chain-ladder method is equivalent to a generalised linear model (GLM) with a log link function relating to the mean of the responses and a Poisson distribution for error structure, $m + a_i + b_j$ as linear predictor. Namely,

Y_{ij} ~Poisson with mean m_{ij}, independently $\forall i, j$

where:

$$\log m_{ij} = m + a_i + b_j$$

and $a_1 = b_1 = 0$

Y_{ij} represents the increment claim amount reported with accident time index i and delay index j.

Renshaw and Verrall (1998) also point out that it is easy to write down a quasi-likelihood, which allows for the variance relationship with the mean to be user specified rather than being fixed according to the distribution function of the error distribution. This will allow the model to be applied to negative incremental claims and the results are always the same as those by the chain-ladder technique when $\sum_{i=1}^{n-j+1} Y_{ij} \geq 0$, where n is the maximum of delay index. They also point out the chain-ladder method will be more appropriate if the run-off triangle consists of claim numbers rather than claim amounts. In the light of this, instead of using Poisson distribution for errors, Gamma, log-normal or inverse-Gaussian distribution may be used for claim size.

In this paper, a gamma distribution is assumed for error distribution, log as the link function, and $m + a_i + b_j$ as linear predictor.

3.3 Asset Classes

In this paper the asset classes are restricted to a short-term fixed interest security represented by 90-day bank bills, long-term fixed interest security represented by 10 year Australian government bonds, and equity represented by the Australian All Ordinaries Index. For simplicity, in the rest of the paper, the three asset classes will be referred to as cash, bond and equity, respectively. It is assumed that cashflows occur at the end of each quarter. Assets are sold at the end of each quarter to pay off the claims that arise during the quarter. After the sale of assets, the weighting of each asset class is assumed to remain the same through the whole ten years. Transaction costs and taxes are ignored in this paper for simplicity.

3.3.1 Stochastic Asset Models

The stochastic asset models examined include Random Walk model, Carter's model, Vector AR(1) model and Harris' Regime Switching Vector AR(1) model. The details of the models are included in the Appendix. For further details, readers may consult the relevant papers included in our reference section.

3.3.2 Return of asset classes

Cash

Since 90-day bank bills are held until redemption date, the return of this short-term security for any quarter t will equal the short-term yields at time t-1 predicted by the stochastic asset model.

Bond

10-year government bonds are assumed to be held until redemption except for those which are sold at the end of each quarter in order to meet claims.

The return of bonds equals $\frac{P_t - P_{t-1} + C_t}{P_{t-1}}$ where P_t represents the price of bonds at time t and C_t the coupon payment during the t^{th} quarter. The half-yearly coupon rate is assumed to be 6% p.a. Since the JC model models the yield of long-term bonds instead of the price of the bond, bond price is calculated based on the yield predicted by the model. Note that the term to maturity of bonds decreases over time. For example, suppose that at the beginning we add 10-year bonds into our asset portfolio, after one quarter the term-to-maturity of these bonds will be 9.75 years. However the JC model only models yield for the short-term security (90 days) and the long-term bond (10-year government bond). In this paper simple linear interpolation is used to obtain the yield corresponding to the term-to-maturity at the end of each quarter. For instance, at the end of the 4th quarter, the term-to-maturity for bonds will reduce to 9 years and suppose that the asset model predicts the short-term yield and long-term yields to be n_4 and l_4, then the yield used to price the bonds at the end of 4th quarter will equal $n_4 + \frac{l_4 - n_4}{10-0.25} \times (9-0.25)$.

Equity

The return of equity will equal the sum of the price yields and dividend yields predicted by the asset model.

3.3.3 Parameter Estimation

The data required for estimation of the parameters of various models and how they are measured are summarised in tables which are included in Appendix. The data used for estimation are the quarterly data series starting from the first quarter of 1981 to the last quarter of 2000.

3.4 Simulation Approach

Since both the asset and liability models adopted in this paper are stochastic, it is intractable to analytically derive the distribution of ultimate surplus for a given asset portfolio. For this reason, a Monte Carlo simulation method is used to approximate the likely distribution of ultimate surplus. While a very wide range of portfolios could be tested in practice, for this worked example, we considered only four. The four are 100% cash, 100% bonds, 100% equities and balanced – 33% cash, 33% bonds, 40% equities.

Apart from future asset returns and future claim payments, the value of the ultimate surplus depends on the value of the initial total asset amount.

The distribution of the ultimate surplus for a portfolio with a given initial asset amount is estimated from the results of 10000 simulations. The probability of a negative ultimate surplus is determined from the number of negative results from the 10,000 simulation results. No account is taken of the size of the negative amounts. All insolvencies are assumed to be "fatal".

4. Results

4.1 Comparison of 'Closest Asset Match' with those derived by using different asset models

The graphs 4.1.1 to 4.1.4 show the relationship between probability of insolvency and initial asset value for the four portfolios examined, if different asset models are used.

Table 4.1 also summarises the "Closest Asset Match" if we use different stochastic asset models and the corresponding approximate lowest initial asset amount required to ensure the probability of insolvency is below 5%.

Table 4.1

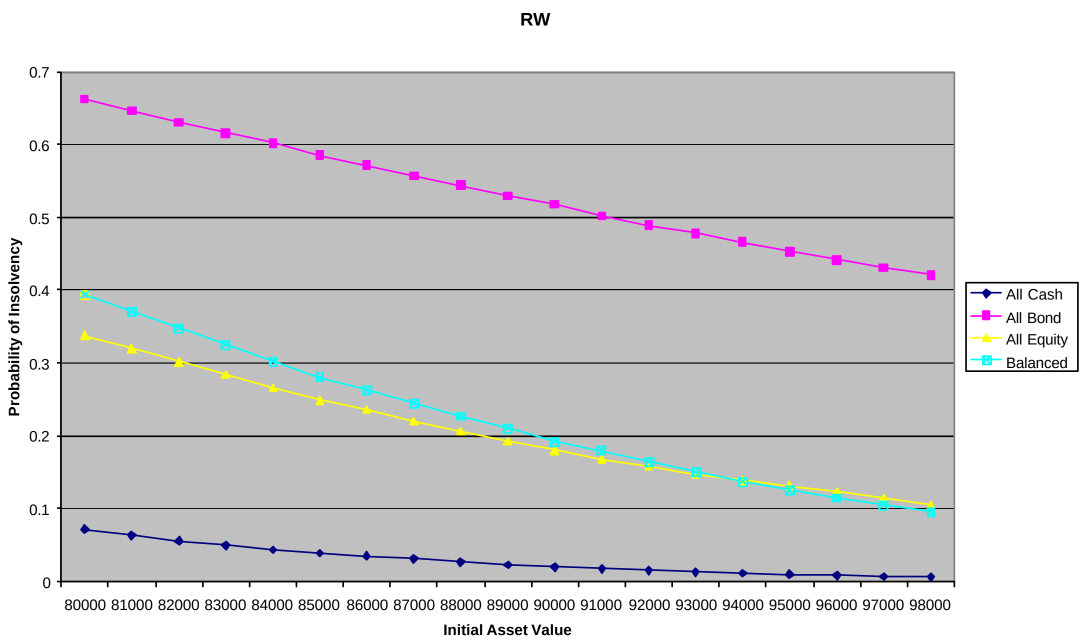
Asset Models	Closet Asset Match	Approximate Initial Asset Required for less than 5% probability of insolvency
RW	All Cash	82000
JC	All Cash	87500
VAR	Balanced	94500
RSVAR	All Cash	90500

The VAR model suggests that the "closest asset match" among the four portfolios examined be the balanced portfolio. This suggests that by increasing the weight of more volatile assets to boost the return of the asset portfolio may decrease the probability of insolvency.

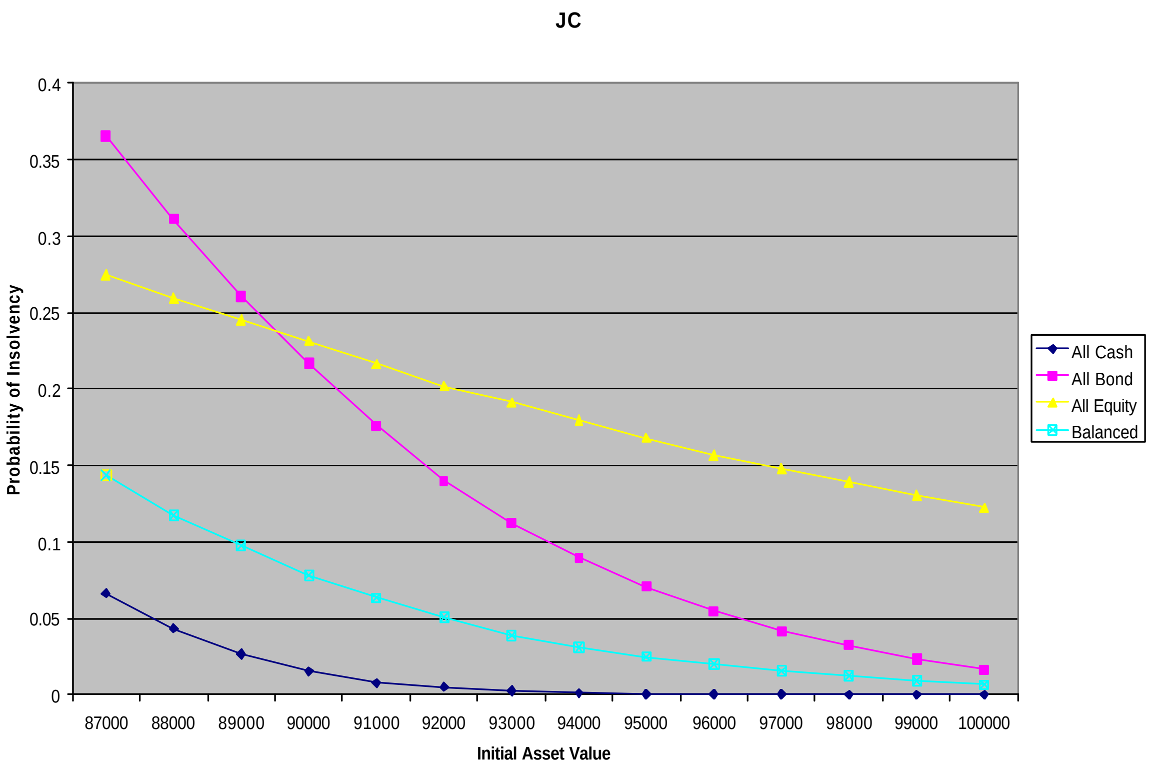
RW, JC mode and RSVAR model all suggest that the "closest asset match" be the all cash portfolio. This is due to the fact the JC model uses a random walk with a drift to model shares and as a result the suggested variance of share price return is large and this makes equity unattractive.

And among the four models, RW model suggests the lowest initial asset value needed to ensure the probability of insolvency below 5%, and VAR suggests the largest number.

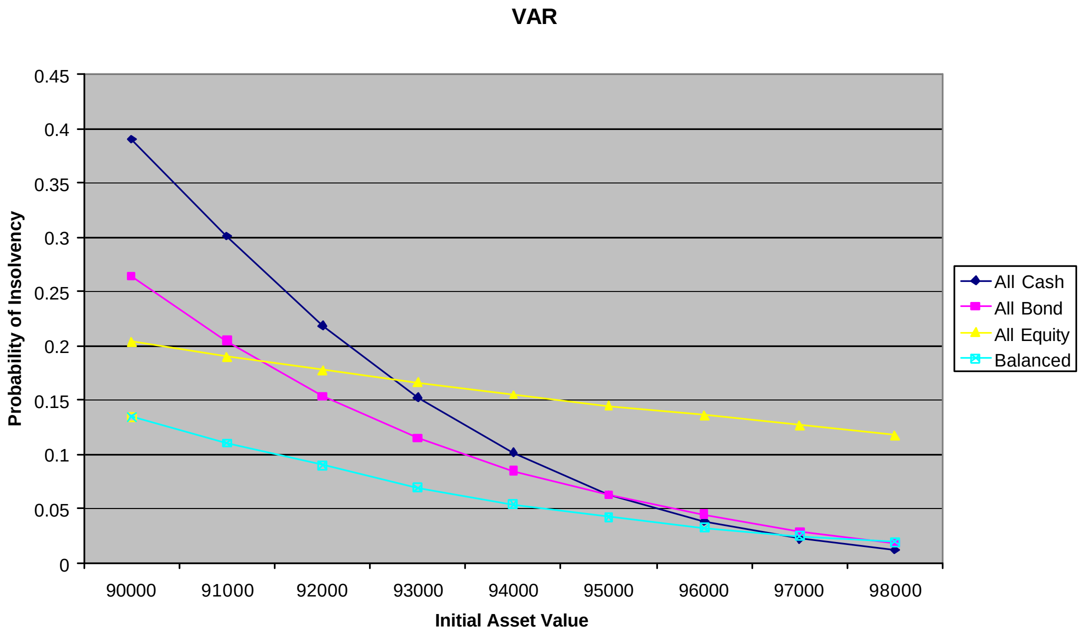
Gragh 4.1.1



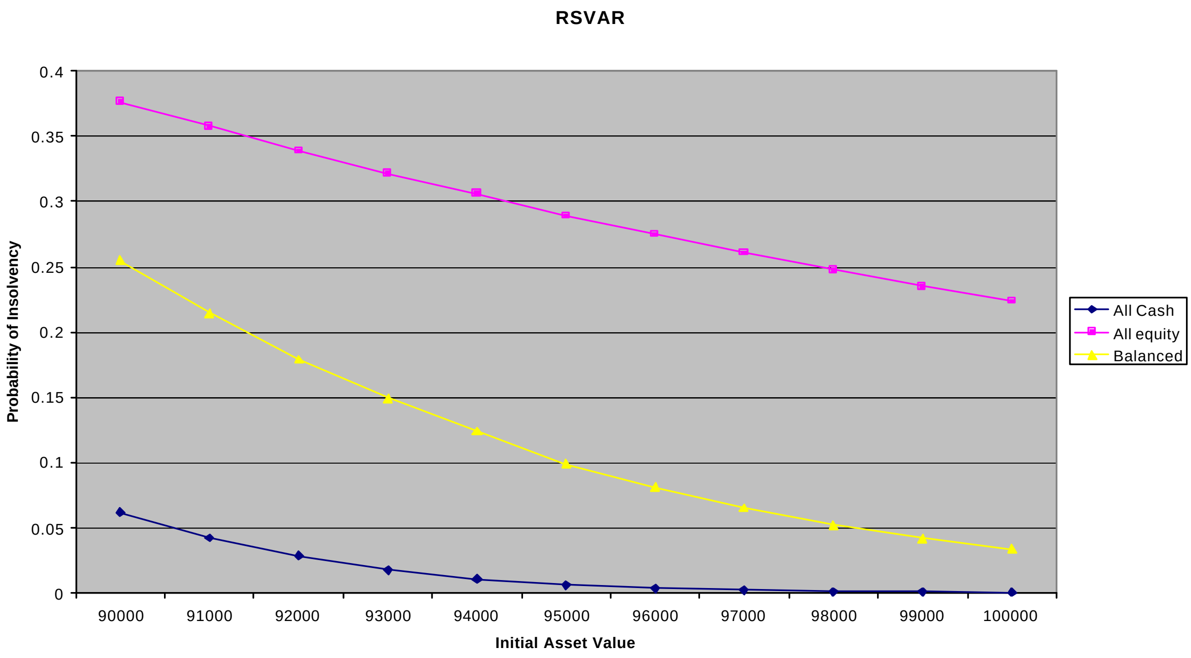
Graph 4.1.2



Graph 4.1.3



Graph 4.1.4



4.2 Comparison of different asset models using in-sample and out-sample data

In this section, RW, JC, VAR(1) and RSVAR(1) models are compared. The comparison is based on the explanatory power of the one-step-ahead forecasts produced by each model. Five variables, namely inflation rate, share price return, share dividend yield, yield of cash, yield of 10-year government bond are compared across asset models. All the variables are measured quarterly and they are all in the form of continuous compounding. Actual values of the five variables are regressed on the forecasts produced by each model. If a model perfectly forecasts the future, we should expect the intercept of the regression line to be 0 and the slope to be 1. R^2 statistics of the regression also shows the proportion of the variance of actual values explained by the forecasts produced by models. Regression analysis is carried out for two periods of data, in-sample data and out-of-sample data. The in-sample data covers the period from the first quarter of 1981 and the first quarter of 2000. The out-of-sample data covers the period form the second quarter of 2000 to the first quarter of 2004. Table 4.2.1[1] summarises the regression results for the in-sample data and Table 4.3.1 summarises the regression results for the out-of-sample data.

Table 4.2.1

Variable		RW	Standard Error	JC	Standard Error	VAR(1)	Standard Error	RSVAR(1)	Standard Error
Inflation Rate	Intercept	0.0048	0.0013	0.0016	0.0014	0.0000	0.0012	0.0009	0.0012
	Slope	**0.5891**	0.0849	**0.8229**	0.0891	**0.9963**	0.0850	**0.9556**	0.0809
	R2	0.3461		0.5355		0.6441		0.6474	
Share Price Return	Intercept	0.0196	0.0117	0.0196	0.0117	-0.0001	0.0166	-0.0033	0.0135
	Slope	0.0000	0.0000	0.0000	0.0000	1.0021	0.6082	**1.1432**	0.3804
	R2	0.0000		0.0000		0.0345		0.1062	
Share Dividend Yield	Intercept	**0.0137**	0.0011	0.0006	0.0015	-0.0012	0.0025	-0.0007	0.0019
	Slope	**-0.3028**	0.1000	**0.9501**	0.1371	**1.1534**	0.2390	**1.1216**	0.1803
	R2	0.0916		0.4036		0.2346		0.3374	
Yield of Cash	Intercept	0.0009	0.0009	0.0013	0.9403	0.0003	0.0011	-0.0008	0.0011
	Slope	**0.9568**	0.0318	**0.9403**	0.0347	**0.9966**	0.0373	**1.0268**	0.0390
	R2	0.9086		0.9120		0.9036		0.9014	
Yield of Bond	Intercept	0.0002	0.0006	0.0002	0.0006	0.0001	0.0074	-0.0003	0.0008
	Slope	**0.9873**	0.0211	**0.9854**	0.0224	**0.9969**	0.0267	**1.0101**	0.0275
	R2	0.9610		0.9645		0.9483		0.9468	

[1] Significant Estimates are bolded

Table 4.2.2

Variable		RW	Standard Error	JC	Standard Error	VAR(1)	Standard Error	RSVAR(1)	Standard Error
Inflation Rate	Intercept	**0.0105**	0.0033	0.0133	0.0059	0.0064	0.0035	0.0068	0.0038
	Slope	-0.1954	0.2744	-0.5011	0.6129	0.4836	0.5603	0.4313	0.6770
	R2	0.0375		0.0489		0.0542		0.0303	
Share Price Return	Intercept	0.0025	0.0150	0.0025	0.0150	-0.0486	0.0285	0.0288	0.0295
	Slope	0.0000	0.0000	0.0000	0.0000	2.0346	1.0000	-0.6812	0.6592
	R2	0.0000		0.0000		0.2415		0.0759	
Share Dividend Yield	Intercept	**0.0113**	0.0023	0.0037	0.0040	-0.0012	0.0025	**0.0088**	0.0028
	Slope	-0.2360	0.2465	0.5763	0.4232	**1.1534**	0.2390	0.0262	0.2726
	R2	0.0658		0.1249		0.2346		0.0007	
Yield of Cash	Intercept	0.0031	0.0022	**0.0088**	0.0015	0.0043	0.0021	0.0037	0.0022
	Slope	**0.7505**	0.1695	**0.3030**	0.1134	**0.6442**	0.1588	**0.6452**	0.1590
	R2	0.6013		0.3545		0.5585		0.5588	
Yield of Bond	Intercept	0.0067	0.5171	**0.0117**	0.0054	**0.0072**	0.00306	0.0064	0.0032
	Slope	**0.9873**	0.2144	**0.5818**	0.1899	**0.4757**	0.2124	**0.5274**	0.2219
	R2	0.0314		0.4192		0.2785		0.3029	

The regression with the in-sample data shows that the explanatory power of the one-step-ahead predictions of the models varies from one variable to another. The result shows that models' one-step-ahead predictions have large explanatory power for yield of cash and yield of bond. None of the models examined has large explanatory power for share price return. Among the four models RSVAR has the largest explanatory power for equity price return with R^2= 10.62%, and RW has the least explanatory power with R^2=0.

Overall the regression with in-sample data suggests that RSVAR fits the data best. And this is expected as RSVAR has more parameters than the other three models.

The regressions with the out-of-sample data shows the explanatory power of the models' one-step-ahead predictions decreases across all five variables except for the VAR(1) model, where R^2 increases from 3.45% to 24%. And some of the slopes are no longer significantly different from zero.

The regression analysis shows that the best fitted model which is RSVAR(1) model may not be the winning model for out-of-sample data. This indicates the importance of judgement when calibrating the parameters. In Australia, there was a major regime shift in the last two decades. In the 1980s we had high inflation and high interest rate and now we have a regime of low inflation and low interest. Any model which assumes this trend will continue infinitely in the future might be proved to be wrong. But a model with parameters estimated based purely on statistical inference from past data will inevitably suggest so.

4.3 Comparison of resulting asset portfolios by using different approaches

We assume the initial asset value we have is 95000 and we use VAR(1) as our stochastic asset model. Table 4.3 summarises the probability of insolvency, mean ultimate surplus, median ultimate surplus and mean squared ultimate surplus, and mean shortfall as defined by Hürlimann (2002), for each portfolio considered.

Table 4.3

Portfolios	Probability of Insolvency	Mean Ultimate Surplus	Median Ultimate Surplus	Mean Squared Ultimate Surplus (closest billion)	Mean Shortfall
All Cash	6.03%	7030	10306	47	-3903
All Bond	6.25%	10369	14821	51	-4156
All Equity	14.41%	87358	65568	190	-10135
Balanced	4.20%	27586	30466	79	-5036

The closest match according to our definition which can be interpreted as that which results in the lowest probability will be the Balanced portfolio. And the closest asset match suggested by Wise (1984), which minimizes the mean squared ultimate surplus, will be the All Cash portfolio. If our objective is to minimize expected shortfall, again the All cash portfolio will be the closest asset match among the four portfolio.

In our opinion the approach suggested by Wise may not be appropriate since under this approach a portfolio which results in a large positive mean surplus will be regarded as undesirable. The mean shortfall reported in the table might have overestimated the mean shortfall which can be realized in real life since some corrective actions such as, put the insurer in administration when it is deemed insolvent, which might prevent the situation getting worse. If we take these corrective actions into account, we should expect the difference of mean shortfalls among the portfolios to be smaller.

The difference between the mean and median suggests that the distributions of ultimate surplus are quite skewed.

Conclusion and Further Research

In this paper, we use different stochastic asset models to identify a "closest" asset match, as defined by Service and Sun (2003), for an assumed portfolio of liability, i.e. the asset portfolio, for a given probability of insolvency, which requires the lowest initial asset amount. It is found that different asset models lead to different "closest" asset match portfolios. This result suggests the importance of selecting the "right" stochastic asset model when identifying the "closest" asset match.

Actual data is regressed on the one-period-ahead forecasts produced by different asset models for both in-sample data, from first quarter of 1981 to the first quarter of 2000, and out-sample data from second quarter of 2000 and first quarter of 2004. It was found that RSVAR(1) best fits the in-sample data, but fails to be the winner for the out-of-sample data.

Finally we find that asset mix derived based on Service and Sun's definition is different from those based on other criteria.

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Appendix 1 Stochastic Asset Models

Random Walk Models

Inflation

$q_t = q_{t-1} + {}^{q}\varepsilon_t$

and $Q_t = Q_{t-1} * \exp(q_t)$

Where dq_t = change in force of inflation over quarter t, happening immediately at the start of quarter t,

q_t = force of inflation per quarter applying over quarter t, from time t-1 to t

Q_t = CPI index at end of quarter t

And, ${}^{q}\varepsilon_t$=i.i.d.N(0, ${}^{q}s^2$)

The model for **short-term yield**

$dn_t = B({}^{n}\omega_1 - {}^{n}\omega_2 B)dq_t + (1 - {}^{n}\theta_3 B^4) * {}^{n}\varepsilon_t$

$n_t = n_{t-1} + dn_t$

$N_t = (\exp(n_t) - 1)*400$

Where, dn_t = change in force of treasury yields over quarter t, happening immediately at the start of quarter t, namely time t-1

n_t = force of treasury yields per quarter applying over quarter t

N_t = Treasury yield over quarter t as % per annum

and, ${}^{n}\omega_t$=i.i.d.N(0,${}^{n}s^2$)

The model for **long-term yield** is

$L_t = L_{t-1} + {}^{l}\varepsilon_t$

Lt = ten year bond yield over quarter t as a nominal per annum rate convertible half yearly

and ${}^{l}\varepsilon_t$=iid N(0, ${}^{l}s^2$),

Share Price

$\rho_t = {}^{\rho}\Phi_0 + {}^{\rho}\varepsilon_t$

$P_t = P_{t-1} * \exp(\rho_t)$

where,

ρ_t = force of share price yields over quarter t, time t-1 to t

P_t = SPI at end of quarter t, time t

and, ${}^{\rho}\varepsilon_t$ = i.i.d.N(0, ${}^{\rho}s^2$),

The model **for share dividends yield**

$Y_t = Y_{t-1} + {}^{y}\varepsilon_t$

Where Y_t= share dividend yield as nominal p.a. convertible quarterly

and, ${}^{y}\varepsilon_t$=i.i.d.N(0, ${}^{y}s^2$).

In addition, it is assumed are correlated and the correlation between them are assumed to be constant for the whole time period examined.

Statistically it is inevitable that the random walk model will produce some negative values for all the variables, this becomes more likely as the variance increases when the time horizon increases. Negative inflation rates and share price returns are economically acceptable; however with share dividend yield, cash yield and bond yield, negative values are unreasonable, therefore a minimum value of 0.5% p.a. is applied to these three variables.

JC Model

The model for inflation is

$dq_t = {}^{q}\Phi_3 dq_{t-1} + (1 - {}^{q}\theta_1 B - {}^{q}\theta_2 B_2) * {}^{q}\varepsilon_t$

and, $q_t = q_{t-1} + dq_t$

and $Q_t = Q_{t-1} * \exp(q_t)$

Where dq_t = change in force of inflation over quarter t, happening immediately at the start of quarter t,

q_t = force of inflation per quarter applying over quarter t, from time t-1 to t

Q_t = CPI index at end of quarter t

And, ${}^{q}\varepsilon_t$=i.i.d.N(0, ${}^{q}s^2$)

The model for **short-term yield**

$$dn_t = B({}^n\omega_1 - {}^n\omega_2 B)dq_t + (1 - {}^n\theta_3 B^4)^* \, {}^n\varepsilon_t$$

$$n_t = n_{t-1} + dn_t$$

$$N_t = (\exp(n_t) - 1)^*400$$

Where, dn_t = change in force of treasury yields over quarter t, happening immediately at the start of quarter t, namely time t-1

n_t = force of treasury yields per quarter applying over quarter t

N_t = Treasury yield over quarter t as % per annum

and, ${}^n\omega_t$=i.i.d.N(0,${}^n s^2$)

The model for **long-term yield** is

$dl_t = {}^l\omega_1 dn_t + {}^l\varepsilon_t$

$l_t = l_{t-1} + dl_t$

$L_t = [\exp(2l_t) - 1] * 200$

and, dl_t= change in force of bond yields over quarter t, happening immediately at start of quarter t, namely time t-1

l_t = force of bond yields over quarter t, from time t-1 to t

Lt = ten year bond yield over quarter t as a nominal per annum rate convertible half yearly

and ${}^1\varepsilon_t$=iid N(0, ${}^l s^2$),

The model for **share price**

$\rho_t = {}^\rho\Phi_0 + {}^\rho\varepsilon_t$

$P_t = P_{t-1} * \exp(\rho_t)$

where,

ρ_t = force of share price yields over quarter t, time t-1 to t

P_t = SPI at end of quarter t, time t

and, ${}^\rho\varepsilon_t$ = i.i.d.N(0, ${}^\rho s^2$),

The model for **share dividends yield and inflation**

$y_t = {}^y\Phi_3 * y_{t-4} + {}^y\Phi_0 * (1 - {}^y\Phi_3) + {}^y\omega_1 q_{t-1} + {}^y\omega_1 \, {}^y\Phi_3 \, q_{t-5} + {}^y\varepsilon_t + {}^y\theta_2 \; {}^y\varepsilon_{t-1}$

$Y_t = [\exp(y_t) - 1] * 400$

Where y_t = force of share dividend yields over quarter t, time t-1 to t

Y_t= share dividend yield as nominal p.a. convertible quarterly

and, ${}^{y}\varepsilon_t$=i.i.d.N(0, ${}^{y}s^2$)

VAR(1) Model

$X_t = M + AX_{t-1} + \xi_t$

$X_t = (G_t, Q_t, \rho_t, \ln Y_t, \ln N_t, \ln L_t)^T$ is a 6*1 column vector of series values at time t.

Where G_t is real Gross Nation Income(GNI) growth,

Q_t is price inflation measured by CPI

ρ_t is share price index returns

$\ln Y_t$ is the logarithm of annual share dividend yields convertiable quarterly

$\ln N_t$ is logarithm short-term interest yields

$\ln L_t$ is logarithm of the yield of ten year government bond.

M is a 6*1 column vector of constants

ξ_t is a 6*1 column vector of independent Normal random errors or shocks to the series at time t. They are not assumed to be contemporaneously correlated.

A is a 6*6 parameter matrix.

RSVAR(1) Model

$X_t = M_{(\rho t)} + A_{(\rho t)}X_{t-1} + \xi_{(\rho t)}z_t$

$Z_t \sim$ i.i.d.N(0,1)

Where ρ_t is defined to be a discrete-valued indicator variable, which indicates the regime that the financial market is in at time t, ρ_t belongs to {1,2}. The transition between regimes is governed by the transition probabilities, $p_{ij}=p(\rho_t=j|\rho_{t-1}=i)$, with $p_{11} + p_{12}=1$ and $p_{21}+p_{22}=1$. The model is therefore a 1st order discrete Markov process.

$X_t = (G_t, Q_t, \rho_t, \ln Y_t, \ln N_t, \ln L_t)^T$ is a 6*1 column vector of series values at time t.

Where G_t is Real Gross Nation Income(GNI) growth,

Q_t is price inflation measured by CPI

ρ_t is share price index returns

lnY_t is the logarithm of annual share dividend yields convertiable quarterly

lnN_t is logarithm short-term interest yields

lnL_t is logarithm of the yield of ten year government bond.

$M_{(\rho t)}$ is a 6*1 column vector of constants for regime $\rho_{t.}$

$\xi_{(\rho t)}$ is a 6*1 column vector of the conditional standard deviations of the variables for regime $\rho_{t.}$

$A_{(\rho t)}$ is a 6*6 parameter matrix for regime $\rho_{t.}$