Multivariate Stop Loss Mixed Erlang Risk: Aggregation, Capital Allocation and Default Risk

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Abstract: In this paper, we address the aggregation of dependent stop loss reinsurance risks where the dependence among the ceding insurer(s) risks is governed by the Sarmanov distribution and each individual risk belongs to the class of Erlang mixtures. We investigate the effects of the ceding insurer(s) risk dependencies on the reinsurer risk profile by deriving a closed formula for the distribution function of the aggregated stop loss reinsurance risk. Furthermore, diversification effects from aggregating reinsurance risks are examined by deriving a closed expression for the risk capital needed for the whole portfolio of the reinsurer and also the allocated risk capital for each business unit under the TVaR capital allocation principle. Moreover, given the risk capital that the reinsurer holds, we express the default probability of the reinsurer analytically. In case the reinsurer is in default, we determine analytical expressions for the amount of the aggregate reinsured unpaid losses and the unpaid losses of each reinsured line of business of the ceding insurer(s). These results are illustrated by numerical examples.

Key words: Risk aggregation; Sarmanov distribution; Mixed Erlang distribution; Capital allocation; Stop loss reinsurance; Reinsurance default risk; Default Probability.

1 Introduction

Reinsurance companies operate in many regions in the world and insure various insurance business lines. In this respect, it is well recognised that the ceding insurer(s) losses are dependent. This risk dependency can be seen between individual risks within each insurance portfolio and also across business lines. Furthermore, the phenomena of dependence also occurs from global risk factors which generate claims simultaneously to each business line, for instance an hurricane damages buildings or cars which affect property lines, at the same time, causes people injuries which influence accident lines. In the risk management framework, for instance the Swiss Solvency Test (SST), similarly to insurance companies, reinsurance companies are obliged to hold a certain level of risk capital in order to be protected from unexpected large losses. The determination of this capital requires the aggregation of the losses generated from each reinsurance portfolio whose distribution depends on the loss distribution of the ceding insurer(s). Meyers et al. [14] is one of the first contribution which have addressed the aggregation of dependent reinsurance risks to evaluate risk capital. In this regard, in order to derive explicit formula for the measure of risk capital including Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) for the aggregated risk, an important task is the appropriate choice of the marginals and the dependence structure between risks. For our framework, mixed Erlang distribution has been chosen as a claim size model for the individual risk of the ceding insurer(s). One of the reason of the tractability of this distribution is the fact that the convolution of such risks belongs again to class of Erlang mixtures, see [9]. Thus stop loss and excess of loss premiums have a closed expression which are very usefull in reinsurance risk modelling, see Lee and Lin [11, 12]. In this contribution, we address the dependence structure between risks by the Sarmanov distribution. The aim of this paper is to analyse the effects of the ceding insurer(s) risk dependencies on the reinsurer risk profile which has only stop loss reinsurance portfolios. Diversification effects from aggregating reinsurance risks

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are examined by deriving a closed expression for the risk capital needed for the whole portfolio and also the allocated risk capital for each business unit. The effects of the reinsurer default are also analysed. The paper is organised as follows: in Section 2 we describe the background of the Sarmanov distribution as a model for the dependence structure between insurance risk and the mixed Erlang distribution with a common scale parameter as a claim size model. The risk model of the ceding insurer is explored in Section 3, with numerical examples, by deriving the joint tail probability of the aggregated risk of two portfolios. In Section 4, the aggregation of stop loss mixed Erlang risks of a reinsurer is addressed by determining a closed form for the distribution function (df) of the aggregated risk. Capital allocation and diversification effects are also presented with numerical studies. We also analyse the default risk of the reinsurer by deriving an analytical form for the expected unpaid losses and the default probability with numerical illustrations. All the proofs are relegated to Section 5. Some properties of the mixed Erlang distribution are presented in the Appendix.

2 Preliminaries

2.1 Sarmanov distribution

Due to its flexibility to model the dependence structure between random variables (rv), the Sarmanov distribution, introduced in Sarmanov [17], have been widely used in many fields. Concerning insurance applications, to calculate Bayes premiums in collective risk model Hernandez et al. [8] have addressed the dependence between risk profiles using multivariate Sarmanov distribution. Sarabia and Gomez [16] have used Sarmanov distribution to fit multivariate insurance count data with Poisson-Beta marginals. The contributions [27, 26] have explored tractable asymptotic formulas in the context of ruin probabilities where the dependence between insurance risks is governed by the Saramanov distribution. Refering to [10], a random vector \((X_1, \ldots, X_n)\) has multivariate Sarmanov distribution with joint density given by

\[
h(x) = \prod_{i=1}^{n} f_i(x_i) \left(1 + \sum_{h=2}^{n} \sum_{1 \leq j_1 < j_2 < \ldots < j_h \leq n} \alpha_{j_1, \ldots, j_h} \prod_{k=1}^{h} \phi_{j_k}(x_{j_k})\right), x := (x_1, \ldots, x_n),
\]

where \(\phi_i\) are kernel functions, which are assumed to be bounded and non-constant such that

\[
E\{\phi_i(X_i)\} = 0,
\]

\[
1 + \sum_{h=2}^{n} \sum_{1 \leq j_1 < j_2 < \ldots < j_h \leq n} \alpha_{j_1, \ldots, j_h} \prod_{k=1}^{h} \phi_{j_k}(x_{j_k}) \geq 0, \quad \forall x_i \in \mathbb{R}
\]

are fulfilled. Some general methods for finding the kernel function \(\phi_i\) was specified by Lee [10] for different types of marginals. In particular, it is commonly used to choose \(\phi_i(x_i) = g_i(x_i) - E\{g_i(X_i)\}\) for marginal distributions with support in \(\mathbb{R}_+\) (see e.g. [26]). The following three cases are the usual specifications of \(g_i(x_i)\):

(i) \(g_i(x_i) = 2F_i(x_i)\) which corresponds to the Farlie-Gumbel-Morgenstern (FGM) distribution, where \(F_i\) is the survival function of \(X_i\),

(ii) \(g_i(x_i) = x_i - E\{X_i^t\}\) such that the \(t\)-th moment \(E\{X_i^t\}\) of \(X_i\) is finite,

(iii) \(g_i(x_i) = e^{-tx_i} - E\{e^{-tX_i}\}\) where \(E\{e^{-tX_i}\} < \infty\) is the Laplace transform of \(X_i\) at \(t\).

2.2 Mixed Erlang Marginals

These last decades, mixed Erlang distribution with a common scale parameter is one of the most usefull model for insurance losses. In risk theory, using the mixed Erlang distribution as a claim size model, an analytical form
for the finite time ruin probability has been derived by Dickson and Willmot [6] and Dickson [5]. Recently, using
the EM algorithm, mixed Erlang distribution has been fitted to catastrophic loss data in the United States by
Lee and Lin [11] and also to censored and truncated data by Verbeleen et al. [21]. Moreover, Lee and Lin [12],
Willmot and Woo [25] have developed the multivariate mixed Erlang distribution to overcome some drawbacks
of the copula approach while Badescu et al. [1] have used multivariate mixed Poisson distribution with mixed
Erlang claim sizes to model operational risks. Furthermore, Hashorva and Ratovomirija [7] have addressed risk
aggregation and capital allocation with mixed Erlang marginals and Sarmanov distribution. In the sequel, we
denote respectively
\[
\begin{align*}
& w_k(x, \beta) = \frac{\beta^k x^k e^{-\beta x}}{(k-1)!}, & W_k(x, \beta) = \sum_{j=k}^{\infty} \frac{(\beta x)^j e^{-\beta x}}{j!}, & \Phi_k(x, \beta) = \sum_{j=0}^{k} \frac{(\beta x)^j e^{-\beta x}}{j!}, & x > 0,
\end{align*}
\]
the pdf, the df and the survival function of an Erlang distribution where \( k \in \mathbb{N}^* \) is the shape parameter and
\( \beta > 0 \) is the scale parameter. As its name indicates, the mixed Erlang distribution is elaborated from the Erlang
distribution, its pdf and df are respectively defined as
\[
\begin{align*}
& f(x, \beta, Q) = \sum_{k=1}^{\infty} q_k w_k(x, \beta), & F(x, \beta, Q) = \sum_{k=1}^{\infty} q_k W_k(x, \beta),
\end{align*}
\]
where \( Q = (q_1, q_2, \ldots) \) is a vector of non-negative weights satisfying \( \sum_{k=1}^{\infty} q_k = 1 \). Hereafter we write \( X \sim ME(\beta, Q) \) if \( X \) has pdf and df given by (2.4). One of the main advantages of the mixed Erlang distribution in
insurance risk modeling is the fact that many useful risk related quantities, such as moments and mean excess
function, have explicit expressions, see e.g., [11, 23, 12, 25]. Furthermore, the mixed Erlang distribution is
a tractable marginal distribution for the Sarmanov distribution. Next we present a result for the correlated
insurance portfolios.

3 Ceding insurance risk model

In this section, we consider two insurance portfolios which both of them consists of \( k \) risks and we denote
\( S_{1,k} = \sum_{i=1}^{k} X_i \) and \( S_{2,k} = \sum_{i=k+1}^{2k} X_i \) the aggregated risk of each portfolio where \( X_i, i = 1, \ldots, 2k \) is a positive
continuous random variable (rv) with finite mean. Hereafter, we assume \( X_i \sim ME(\beta_i, Q_i), i = 1, \ldots, 2k \) and the
dependence structure between risks within and across the portfolio is governed by the Sarmanov distribution
with kernel function
\[
\phi_i(x_i) = g_i(x_i) - E(g_i(X_i)),
\]
which shall be abbreviated as
\[
(X_{1,1}, \ldots, X_{2k}) \sim SME(\beta, Q)
\]
where \( \beta = (\beta_1, \ldots, \beta_{2k}) \), \( Q = (Q_1, \ldots, Q_{2k}) \). In the rest of the paper we consider for \( g_i \) one of the three cases
described in (i), (ii) and (iii).

Furthermore, we define two vectors of mixing weights \( \Theta(Q_i) \) and \( \Psi(Q_i) \) where their components depend on
the kernel function \( \phi_i \). In particular, the components of \( \Theta(Q_i) = (\theta_{i,1}, \theta_{i,2}, \ldots) \) are given by:

- for \( g_i(x_i) = 2F_i(x_i) \), \( \theta_{i,s} = \frac{1}{2^s} \sum_{j=1}^{k} (s-1) \sum_{l=s-j+1}^{\infty} q_{i,l} s, s = 1, 2, \ldots \),
- for \( g_i(x_i) = x_i^t \),
\[
\theta_{i,s} = \begin{cases} 
0 & \text{for } s \leq t, \\
\frac{q_{i,s-1}}{\sum_{j=1}^{t} q_j} \frac{1}{F^{(t)}} & \text{for } s > t,
\end{cases}
\]
\[ \text{for } g_i(x_i) = e^{-tx_i}, \theta_{i,s} = \frac{q_{i,s} \beta'}{\sum_{j=1}^{n} q_{i,j} \beta'} \text{ with } \beta = \frac{\beta}{\beta+1}, s = 1, 2, \ldots, \]

whilst the components of \( \Psi(Q) = (\psi_{i,1}, \psi_{i,2}, \ldots) \) are given by

\[ \psi_{i,s} = \sum_{j=1}^{s} q_{i,j} \left( k - 1 \right) \left( \frac{\beta_j}{Z(\beta_{2k})} \right) \left( 1 - \frac{\beta_j}{Z(\beta_{2k})} \right)^{s-j}, \]

where \( Z(\beta_{2k}) = 2 \beta_{2k} \) for \( g_i(x_i) = 2F(x_i) \), \( Z(\beta_{2k}) = \beta_{2k} \) for \( g_i(x_i) = x_i^t \) and \( Z(\beta_{2k}) = \beta_{2k} + t \) for \( g_i(x_i) = e^{-tx_i} \).

Moreover, for given mixing weights \( V_i = (v_{1i}, v_{2i}, \ldots), i = 1, \ldots, n + 1 \) we define a vector of mixing probability \( \Pi(V_1, \ldots, V_{n+1}) \) as follows

\[ \pi_l\{V_1, \ldots, V_{n+1}\} = \left\{ \begin{array}{ll}
0 & \text{for } l = 1, \ldots, n, \\
\sum_{j=1}^{l-1} \pi_j\{V_1, \ldots, V_n\} v_{n+1,l-j} & \text{for } l = n + 1, n + 2, \ldots.
\end{array} \right. \]

We present next the main result of this section.

**Proposition 3.1** If \( (X_1, \ldots, X_{2k}) \sim SME(\beta, Q) \) with \( \beta_{2k} \geq \beta, i = 1, \ldots, 2k - 1, \) then the joint tail probability of \( S_{1,k} \) and \( S_{2,k} \) is given by (set \( \gamma_{jm} := \mathbb{E}\{g_{jm}(X_{jm})\} \))

\[ P(S_{1,k} > u_1, S_{2,k} > u_2) = \xi_1 F_{S_{1,k}}(u_1) F_{S_{2,k}}(u_2) + \sum_{i=1}^{2k} \sum_{j_1, j_2} \prod_{m=1}^{l} \gamma_{jm} F_{S_{1,k}}(u_1) F_{S_{2,k}}(u_2), \]

where

\[ \xi_1 = 1 + \sum_{j_1} \sum_{j_2} \alpha_{j_1,j_2} \gamma_{j_1} \gamma_{j_2} - \sum_{j_1} \sum_{j_2} \sum_{j_3} \alpha_{j_1,j_2,j_3} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} + \ldots + \left( -1 \right)^{2k} \alpha_{1,\ldots,2k} \prod_{j=1}^{2k} \gamma_j, \]

\[ \xi_{j_1,j_2} = \sum_{j_3} \left( - \sum_{j_2} \alpha_{j_1,j_2} \gamma_{j_2} + \sum_{j_2} \sum_{j_3} \alpha_{j_1,j_2,j_3} \gamma_{j_2} \gamma_{j_3} + \ldots + (-1)^{2k+1} \alpha_{1,\ldots,2k} \prod_{i \in C\{j_1\}} \gamma_i \right), \]

\[ \xi_{j_1,j_2,j_3} = \sum_{j_4} \sum_{j_2} \left( \alpha_{j_1,j_2,j_3} \gamma_{j_3} + \sum_{j_2} \sum_{j_4} \alpha_{j_1,j_2,j_3,j_4} \gamma_{j_3} \gamma_{j_4} + \ldots + \left( -1 \right)^{2k+1} \alpha_{1,\ldots,2k} \prod_{i \in C\{j_1,j_2\}} \gamma_i \right), \]

\[ \xi_{j_1,j_2,k-1} = \sum_{j_4} \ldots \sum_{j_{2k-1}} \alpha_{j_1,\ldots,j_{2k-1}} - \alpha_{1,\ldots,2k} \gamma_{j_{2k}}, \]

\[ \xi_{j_1,\ldots,j_{2k}} = \alpha_{1,\ldots,2k}, \]

with \( C = \{1, \ldots, 2k\}, j_1 \in C, j_2 \in C\setminus\{j_1\}, j_3 \in C\setminus\{j_1,j_2\}, \ldots, j_{2k} \in C\setminus\{j_1,\ldots,j_{2k-1}\}, \)

\[ S_{1,k} \sim ME(\Pi(\Psi(Q_1), \ldots, \Psi(Q_k), Z(\beta_{2k}))), \]

\[ S_{2,k} \sim ME(\Pi(\Psi(Q_{k+1}), \ldots, \Psi(Q_{2k}), Z(\beta_{2k}))), \]

\[ \tilde{S}_{1,k} \sim ME(\Pi(\Psi(Q_1^t), \ldots, \Psi(Q_k^t), Z(\beta_{2k}))), \]

\[ \tilde{S}_{2,k} \sim ME(\Pi(\Psi(Q_{k+1}^t), \ldots, \Psi(Q_{2k}^t), Z(\beta_{2k}))), \]

and for \( i = 1, \ldots, 2k \)

\[ Q_i^t = \begin{cases} Q_i & \text{if } i \notin \{j_1, j_2, \ldots, j_l\}, \\
\Theta(Q_i) & \text{if } i \in \{j_1, j_2, \ldots, j_l\}. \end{cases} \]

**Example 3.2** Assume that the ceding insurer has two portfolios say Portfolio A and Portfolio B. Concerning the dependence structure between risks, two cases of kernel function are considered \( \phi(x_i) = 2F_i(x_i) - 1 \) which
defines the FGM distribution as explored in [2] and \( \phi_i(x_i) = e^{-x_i} - \mathbb{E}\{e^{-X_i}\} \) introduced by Hashorva and Ratovomirija [7] for mixed Erlang marginals. In the rest of the paper we refer to the latter as the Laplace case. Table 3.1 presents the parameters of each individual risk \( X_i, i = 1, \ldots, 4 \) and their central moments, whilst Table 3.2 displays the dependence parameters between \( X_1, X_2, X_3 \) and \( X_4 \). We note that these dependence parameters have been chosen so that (2.2) holds.

<table>
<thead>
<tr>
<th>Portfolio A</th>
<th>( X_1 )</th>
<th>( \beta_1 )</th>
<th>( Q_1 )</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.12</td>
<td>(0.4, 0.6)</td>
<td>13.33</td>
<td>127.78</td>
<td>1.55</td>
<td>6.50</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>(0.3, 0.7)</td>
<td>12.14</td>
<td>97.45</td>
<td>1.49</td>
<td>4.33</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio B</th>
<th>( X_3 )</th>
<th>( \beta_1 )</th>
<th>( Q_1 )</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.15</td>
<td>(0.5, 0.5)</td>
<td>10.00</td>
<td>77.78</td>
<td>1.62</td>
<td>6.80</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.16</td>
<td>(0.8, 0.2)</td>
<td>7.50</td>
<td>53.13</td>
<td>1.88</td>
<td>8.16</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Parameters and central moments of \( X_i, i = 1, 2, 3, 4 \).

<table>
<thead>
<tr>
<th></th>
<th>( \alpha_{1,2} )</th>
<th>( \alpha_{1,3} )</th>
<th>( \alpha_{1,4} )</th>
<th>( \alpha_{2,3} )</th>
<th>( \alpha_{2,4} )</th>
<th>( \alpha_{3,4} )</th>
<th>( \alpha_{1,2,3} )</th>
<th>( \alpha_{1,2,4} )</th>
<th>( \alpha_{1,3,4} )</th>
<th>( \alpha_{2,3,4} )</th>
<th>( \alpha_{1,2,3,4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>FGM</td>
<td>0.6</td>
<td>0.1</td>
<td>0.1</td>
<td>0.04</td>
<td>0.5</td>
<td>0.11</td>
<td>0.12</td>
<td>0.10</td>
<td>0.15</td>
<td>0.07</td>
<td></td>
</tr>
<tr>
<td>Laplace</td>
<td>16</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>8</td>
<td>5</td>
<td>30</td>
<td>15</td>
<td>20</td>
<td>170</td>
</tr>
</tbody>
</table>

Table 3.2: Dependence parameters of \((X_1, X_2, X_3, X_4)\).

It can be seen from Table 3.3 that the interdependence between the two insurance portfolios yields high probability for the aggregated risk of each portfolio to exceed simultaneously some threshold.

<table>
<thead>
<tr>
<th>Thresholds</th>
<th>Independence case</th>
<th>Laplace case</th>
<th>FGM case</th>
</tr>
</thead>
<tbody>
<tr>
<td>((u_1, u_2))</td>
<td>( P(S_{1,2} &gt; u_1, S_{2,2} &gt; u_2) )</td>
<td>( P(S_{1,2} &gt; u_1, S_{2,2} &gt; u_2) )</td>
<td>( P(S_{1,2} &gt; u_1, S_{2,2} &gt; u_2) )</td>
</tr>
<tr>
<td>((20,15))</td>
<td>0.1494</td>
<td>0.1569</td>
<td>0.1573</td>
</tr>
<tr>
<td>((25,20))</td>
<td>0.0697</td>
<td>0.0751</td>
<td>0.0795</td>
</tr>
<tr>
<td>((30,25))</td>
<td>0.0304</td>
<td>0.0331</td>
<td>0.0374</td>
</tr>
<tr>
<td>((35,30))</td>
<td>0.0125</td>
<td>0.0138</td>
<td>0.0165</td>
</tr>
</tbody>
</table>

Table 3.3: Joint tail probability of \( S_{1,2} = X_1 + X_2 \) and \( S_{2,2} = X_3 + X_4 \).

4 Reinsurance risk model

In this section, we denote \( R_2 := T_{1,k} + T_{2,k} \) the aggregate reinsurance stop loss risk where \( T_{1,k} := (S_{1,k} - d_1)_+ \) and \( T_{2,k} := (S_{2,k} - d_2)_+ \) represent two stop loss reinsurance portfolios with \( S_{1,k} = \sum_{i=1}^{k} X_i \) and \( S_{2,k} = \sum_{i=k+1}^{2k} X_i \) the ceding insurer aggregated risk and \( d_i, i = 1, 2 \) some positive deductible. Additionally, for a given risk \( X \sim ME(\beta, Q) \) with df F and for a deductible \( d > 0 \) we denote in the rest of the paper

\[
F_X(y + d) = \sum_{k=0}^{\infty} \Delta_k(d, \beta, Q)W_{k+1}(y, \beta),
\]

\[
U_X(c, d, \beta) = \sum_{k=0}^{\infty} (k + 1)\Delta_k(d, \beta, Q)W_{k+2}(c, \beta),
\]

with

\[
\Delta_k(d, \beta, Q) = \frac{1}{\beta} \sum_{j=0}^{\infty} q_{j+k+1}w_{j+1}(d, \beta).
\]
Furthermore, for $X_i \sim ME(Q_i, \beta_i)$, with $d_i > 0, i = 1, 2$ we define

$$F_{X_1+X_2}(d_1, d_2, s) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \Delta_k(d_1, \beta, Q_1) \Delta_j(d_2, \beta, Q_2) W_{k+j+2}(s, \beta),$$

$$U_{X_1}(c, d_1, d_2, \beta) = \frac{1}{\beta} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k + 1) \Delta_k(d_1, \beta, Q_1) \Delta_j(d_2, \beta, Q_2) W_{k+j+3}(c, \beta),$$

$$U_{X_2}(c, d_1, d_2, \beta) = \frac{1}{\beta} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (j + 1) \Delta_k(d_1, \beta, Q_1) \Delta_j(d_2, \beta, Q_2) W_{k+j+3}(c, \beta),$$

$$U_{X_1+X_2}(c, d_1, d_2, \beta) = \frac{1}{\beta} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k + j + 2) \Delta_k(d_1, \beta, Q_1) \Delta_j(d_2, \beta, Q_2) W_{k+j+3}(c, \beta).$$

### 4.1 Aggregation of reinsurance stop loss risks

In the following result we show that the df of the aggregated stop loss risk $R_2$ has a closed form which allows us to derive analytical formula of its mean excess function.

**Proposition 4.1** If $(X_1, \ldots, X_{2k}) \sim SME(\beta, Q)$ with $\beta_{2k} \geq \beta_i, i = 1, \ldots, 2k-1$ and $d_j > 0, j = 1, 2$, then the df of the aggregated stop loss risk $R_2$ is given by

$$F_{R_2}(s) = \begin{cases} F_{S_{1,k},S_{2,k}}(d_1, d_2) & \text{for } s = 0, \\ F_{S_{1,k},S_{2,k}}(d_1 + s, d_2 + s) & \text{for } s > 0, \end{cases}$$

where

$$F_{S_{1,k},S_{2,k}}(d_1, d_2) = \xi_1 F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2) + \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \xi_{j_1, j_2, \ldots, j_l} \prod_{m=1}^{l} \gamma_m F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2),$$

$$F_{S_{1,k},S_{2,k}}(d_1 + s, d_2 + s) = \xi_1 \left( F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2 + s) + F_{S_{1,k}}(d_1 + s) F_{S_{2,k}}(d_2) + F_{S_{1,k} + S_{2,k}}(d_1, d_2, s) \right)$$

$$+ \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \xi_{j_1, j_2, \ldots, j_l} \prod_{m=1}^{l} \gamma_m \left( F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2 + s) \right)$$

$$+ F_{S_{1,k}}(d_1 + s) F_{S_{2,k}}(d_2) + F_{S_{1,k} + S_{2,k}}(d_1, d_2, s),$$

with $S_{1,k}, S_{2,k}, \tilde{S}_{1,k}, \tilde{S}_{2,k}$ are defined in Proposition 3.1.

**Remarks 4.2** Given the tractable form of the df in (4.1), many risk related quantities for $R_2$ have an explicit form, for instance, for $c > 0$ the mean excess function of $R_2$ is given by

$$E \{ R_2 - c | R_2 > c \} = \frac{1}{F_{R_2}(c)} \left[ \xi_1 \left( F_{S_{1,k}}(d_1) U_{S_{2,k}}(c, d_2, Z(\beta_{2k})) + F_{S_{2,k}}(d_2) U_{S_{1,k}}(c, d_1, Z(\beta_{2k})) \right) 

+ U_{S_{1,k} + S_{2,k}}(c, d_1, d_2, Z(\beta_{2k})) + \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \xi_{j_1, j_2, \ldots, j_l} \prod_{m=1}^{l} \gamma_m \left( F_{S_{1,k}}(d_1) U_{S_{2,k}}(c, d_2, Z(\beta_{2k})) \right) 

+ F_{S_{2,k}}(d_2) U_{S_{1,k}}(c, d_1, Z(\beta_{2k})) + U_{S_{1,k} + S_{2,k}}(c, d_1, d_2, Z(\beta_{2k})) \right] - c.$$ (4.2)

**Example 4.3** In this illustration, we consider the same parameters of each individual risk of the ceding insurer portfolios as in Table 3.1. Furthermore, we assume that the ceding insurer re-insures its two portfolios to a
reinsurer with stop loss programs where the deductibles are \( d_1 = 40 \) and \( d_2 = 30 \) for Portfolio A and for Portfolio B, respectively. In practice, it is recognised that risk measures on the aggregated risk are sensitive to the strength of the dependence between individual risks. Actually, by taking into account the dependence within and across the ceding insurer portfolios which is determined by the parameters in Table 3.2, the aggregated risk \( R_2 \) of the reinsurer is riskier than in the independence case. Therefore, based on VaR and TVaR as a risk measure, the reinsurer needs much more risk capital in the dependence case. Furthermore, for a different confidence level \( p \), it can be seen that the deviation from the independence assumption is greater for VaR than for TVaR.

\[
D_p(T_1, T_2) = 1 - \frac{\text{TVaR}_{R_2}(p)}{\text{TVaR}_{T_1}(p) + \text{TVaR}_{T_2}(p)}.
\]

As presented in Table 4.2, diversification benefits increase with the confidence level. Conversely, the deviation from the independence case yields a reduction of the diversification benefits which is obvious since the full diversification effects are attained when risks are independent.

<table>
<thead>
<tr>
<th>Confidence level</th>
<th>Independence case</th>
<th>Laplace case</th>
<th>FGM case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( \text{VaR}_{R_2}(p) )</td>
<td>( \text{TVaR}_{R_2}(p) )</td>
<td>( \text{VaR}_{R_2}(p) )</td>
</tr>
<tr>
<td>90.00 %</td>
<td>11.73</td>
<td>22.64</td>
<td>11.98</td>
</tr>
<tr>
<td>92.50 %</td>
<td>14.98</td>
<td>25.76</td>
<td>15.24</td>
</tr>
<tr>
<td>95.00 %</td>
<td>19.47</td>
<td>30.10</td>
<td>19.75</td>
</tr>
<tr>
<td>97.50 %</td>
<td>26.97</td>
<td>37.40</td>
<td>27.27</td>
</tr>
<tr>
<td>99.00 %</td>
<td>36.64</td>
<td>46.85</td>
<td>36.97</td>
</tr>
<tr>
<td>99.50 %</td>
<td>43.80</td>
<td>53.89</td>
<td>44.15</td>
</tr>
<tr>
<td>99.90 %</td>
<td>60.08</td>
<td>69.92</td>
<td>60.45</td>
</tr>
</tbody>
</table>

Table 4.1: Deviation of VaR and TVaR from the independence case.

It is well known that risk diversification across portfolios arises from aggregating their individual risks, see e.g., [20], [18]. In this respect, by considering the TVaR as a measure for the risk capital, the diversification benefits \( D_p(T_1, T_2) \) are quantified as the relative reduction of the risk capital required for the whole portfolio of the reinsurer from aggregating the stop loss risk \( T_1 \) and \( T_2 \) as follows

\[
D_p(T_1, T_2) = 1 - \frac{\text{TVaR}_{R_2}(p)}{\text{TVaR}_{T_1}(p) + \text{TVaR}_{T_2}(p)}.
\]

| \( p \)          | \( \text{TVaR}_{R_2}(p) \) | \( \text{TVaR}_{T_1}(p) \) | \( \text{TVaR}_{T_2}(p) \) | \( D_p(T_1, T_2) \) (%)
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Independence case</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95.00 %</td>
<td>30.10</td>
<td>24.87</td>
<td>18.26</td>
<td>30.19</td>
</tr>
<tr>
<td>97.50 %</td>
<td>37.40</td>
<td>32.34</td>
<td>24.26</td>
<td>33.92</td>
</tr>
<tr>
<td>99.00 %</td>
<td>46.85</td>
<td>41.89</td>
<td>31.97</td>
<td>36.56</td>
</tr>
<tr>
<td>99.90 %</td>
<td>69.92</td>
<td>64.84</td>
<td>50.55</td>
<td>39.40</td>
</tr>
<tr>
<td>Laplace case</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95.00 %</td>
<td>30.41</td>
<td>25.11</td>
<td>18.41</td>
<td>30.13</td>
</tr>
<tr>
<td>97.50 %</td>
<td>37.73</td>
<td>32.58</td>
<td>24.42</td>
<td>33.82</td>
</tr>
<tr>
<td>99.00 %</td>
<td>47.21</td>
<td>42.14</td>
<td>32.13</td>
<td>36.44</td>
</tr>
<tr>
<td>99.90 %</td>
<td>70.31</td>
<td>65.07</td>
<td>50.72</td>
<td>39.28</td>
</tr>
<tr>
<td>FGM case</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95.00 %</td>
<td>33.14</td>
<td>27.71</td>
<td>18.64</td>
<td>28.52</td>
</tr>
<tr>
<td>97.50 %</td>
<td>40.68</td>
<td>35.40</td>
<td>24.69</td>
<td>32.29</td>
</tr>
<tr>
<td>99.00 %</td>
<td>50.40</td>
<td>45.14</td>
<td>32.44</td>
<td>35.03</td>
</tr>
<tr>
<td>99.90 %</td>
<td>73.89</td>
<td>68.32</td>
<td>51.08</td>
<td>38.11</td>
</tr>
</tbody>
</table>

Table 4.2: Diversification benefits based on TVaR of the aggregate risk \( R_2 \) and the individual risk \( T_i, i = 1, 2 \).
4.2 TVaR capital allocation

In this section, we derive analytical expressions for the amount of capital allocated to each individual risk of the reinsurer under the TVaR principle. In the enterprise risk management framework, to absorb large unexpected losses, reinsurers are required to hold a certain amount of economic capital for the entire portfolio. In this respect, the so-called capital allocation consists in attributing the required capital to each individual line. This allows the reinsurance company to identify and to monitor efficiently their risks. In the literature, many capital allocation techniques have been developed, see for instance [3, 19, 20, 4, 13] and references therein. In practice, it is well known that the TVaR principle takes into account the dependence structure between risks and satisfy the full allocation principle. More precisely, if \( R_n = \sum_{i=1}^{n} T_i \) is the aggregate risk where \( T_i \) is a rv with finite mean that represents the individual risk of the reinsurer, the amount of capital \( TVaR_p(T_i, R_n) \) required for each risk \( T_i \), for \( i = 1, \ldots, n \), is defined as

\[
TVaR_p(T_i, R_n) = \frac{\mathbb{E}(T_i \mathbb{1}_{\{R_n > VaR_{R_n}(p)\}})}{1 - p}, \tag{4.3}
\]

where \( p \in (0, 1) \) is the tolerance level. The full allocation principle implies

\[
TVaR_{R_n}(p) = \sum_{i=1}^{n} TVaR_p(T_i, R_n)
\]

which means that, based on TVaR as a risk measure, the capital required for the entire portfolio is equal to the sum of the required capital of each risk within the portfolio. The following proposition develops an explicit form for \( TVaR_p(T_i, R_2) \), \( i = 1, 2 \), in the case of stop loss mixed Erlang type risks. In addition, we define below \( S_{1,k}, S_{2,k}, \tilde{S}_{1,k}, \tilde{S}_{2,k} \) as in Proposition 4.1 and we denote \( x_p := VaR_{R_2}(p) \).

**Proposition 4.4** Let \( (X_1, \ldots, X_{2k}) \sim SME(\beta, Q) \) with \( \beta_{2k} \geq \beta_i, i = 1, \ldots, 2k - 1 \) and \( d_j > 0, j = 1, 2 \). If further \( T_j, j = 1, 2 \) has finite mean then

\[
TVaR_p(T_1, R_2) = \frac{1}{1 - p} \left[ \xi_1 \left( F_{\tilde{S}_{2,k}}(d_2) \mathcal{U}_{S_{1,k}}(x_p, d_1, Z(\beta_{2k})) + \mathcal{U}_{S_{1,k}}(x_p, d_1, d_2, Z(\beta_{2k})) \right) \right. \\
+ \sum_{i=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \xi_{j_1, j_2, \ldots, j_l} \prod_{m=1}^{l} \gamma_{j_m} \left( F_{\tilde{S}_{2,k}}(d_2) \mathcal{U}_{S_{1,k}}(x_p, d_1, Z(\beta_{2k})) \right) \\
+ \left. \mathcal{U}_{\tilde{S}_{1,k}}(x_p, d_1, d_2, Z(\beta_{2k})) \right].
\]

**Example 4.5** In this example, we consider the same individual risks and dependence parameters as in Example 3.2 and the reinsurance programs as in Example 4.3. Based on TVaR as a risk measure for quantifying the risk capital required for the whole portfolio the required capital of each stop loss risk \( T_i, i = 1, 2 \) are evaluated for different confidence level \( p \). Since \( T_1 \) is riskier than \( T_2 \), as displayed in Table 4.3, more capital is required for \( T_1 \) compared to the amount needed for \( T_2 \).
Table 4.3: TVaR and allocated capital to each stop loss risk $T_i, i = 1, 2$, under the TVaR capital allocation principle.

<table>
<thead>
<tr>
<th>$p$</th>
<th>TVaR$_{R_1}(p)$</th>
<th>TVaR$_{R_1}(T_1, R_2)$</th>
<th>TVaR$_{R_2}(p)$</th>
<th>TVaR$_{R_1}(T_1, R_2)$</th>
<th>TVaR$_{R_2}(T_2, R_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>90.00%</td>
<td>22.93</td>
<td>14.56</td>
<td>8.37</td>
<td>25.35</td>
<td>16.19</td>
</tr>
<tr>
<td>92.50%</td>
<td>26.06</td>
<td>16.85</td>
<td>9.21</td>
<td>28.62</td>
<td>18.62</td>
</tr>
<tr>
<td>95.00%</td>
<td>30.41</td>
<td>20.15</td>
<td>10.26</td>
<td>33.14</td>
<td>22.12</td>
</tr>
<tr>
<td>97.50%</td>
<td>37.73</td>
<td>25.99</td>
<td>11.74</td>
<td>40.68</td>
<td>28.21</td>
</tr>
<tr>
<td>99.00%</td>
<td>47.21</td>
<td>33.94</td>
<td>13.27</td>
<td>50.40</td>
<td>36.39</td>
</tr>
<tr>
<td>99.50%</td>
<td>54.25</td>
<td>40.03</td>
<td>14.22</td>
<td>57.59</td>
<td>42.59</td>
</tr>
<tr>
<td>99.90%</td>
<td>70.31</td>
<td>54.20</td>
<td>16.11</td>
<td>73.89</td>
<td>56.79</td>
</tr>
</tbody>
</table>

4.3 Reinsurer default analysis

In the enterprise risk management framework, reinsurers are obliged to hold a certain amount of capital $K > 0$ in order to be covered from unexpected large losses. The amount of this capital is determined so that the reinsurer will be able to honor its liabilities even in the worst case with high probability. For instance, in the SST, $K$ is quantified as the TVaR at a tolerance level 99% of the aggregated risk $R_n = \sum_{i=1}^{n} T_i$ where $T_i$ represents the individual risk of the reinsurer. This means that for 99% probability the reinsurer has enough buffer to pay its obligations. However, in case $R_n > K$ the reinsurer is in default and thus ceding insurers are not protected from losses exceeding $K$ i.e. $R_n - K$. By analogy to the case between the insurer and the policyholders, see [15], the quantity $(R_n - K)_+$ is called the default option of the reinsurer or in other words the ceding insurers deficit with $U(K) := \mathbb{E} \left\{ (R_n - K)_+ \right\}$ the value of the default option. In view of the full capital allocation principle, for a given risk capital $K$ required for the entire portfolio of the reinsurer, if $K_i, i = 1, \ldots, n$ is the risk capital needed for each individual risk then $K = \sum_{i=1}^{n} K_i$. Furthermore, the value of the default option is also defined as the sum of the value of the unpaid losses $U(K_i, K) := \mathbb{E} \left\{ (T_i - K_i) \mathbb{1}_{\{R_n > K\}} \right\}$ of each ceding insurer(s) reinsured lines of business, specifically (see e.g. [4])

$$U(K) = \sum_{i=1}^{n} U(K_i, K)$$

Next, we determine $U(K)$ and also derive an explicit formula for the default probability of the reinsurer $\phi(K) := \mathbb{P}(R_2 > K)$ where $R_2 = T_1 + T_2$. Furthermore, given that the reinsurer is in default, analytical expressions of the unpaid excess losses of each line of business of the ceding insurer(s) are derived.

**Proposition 4.6** If $(X_1, \ldots, X_{2k}) \sim SME(\beta, Q)$ with $\beta_{2k} \geq \beta_i, i = 1, \ldots, 2k - 1$ and $d_j > 0, j = 1, 2$ then for a given risk capital $K > 0$

$$U(K) = \xi_1 \left( F_{S_{1,k}(d_1)} U_{S_{1,k}}(K, d_2, Z(\beta_{2k})) + F_{S_{2,k}(d_2)} U_{S_{1,k}}(K, d_1, Z(\beta_{2k})) ight) + \sum_{l=1}^{2k} \sum_{j_1,j_2,\ldots,j_l} \gamma_{j_1,j_2,\ldots,j_l} \left( F_{S_{1,k}(d_1)} U_{S_{2,k}}(K, d_2, Z(\beta_{2k})) + F_{S_{2,k}(d_2)} U_{S_{1,k}}(K, d_1, Z(\beta_{2k})) \right) - K \mathcal{F}_{R_2}(K),$$

where $\mathcal{F}_{R_2}(K) = 1 - F_{R_2}(K)$ with $F_{R_2}(\cdot)$ is defined in Proposition 4.1.
Remarks 4.7 In view of Proposition 4.1 analytical expression for the default probability of the reinsurer is given by

\[
\phi(K) = 1 - \xi_1 \left( F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2 + K) + F_{S_{1,k}}(d_1 + s) F_{S_{2,k}}(d_2) + F_{S_{1,k}+S_{2,k}}(d_1, d_2, K) \right) \\
+ \sum_{l=1}^{2k} \sum_{j=1}^{2k} \xi_{j_1,j_2,...,j_l} \prod_{m=1}^{l} \gamma_{jm} \left( F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2 + K) \right) \\
+ F_{S_{1,k}}(d_1 + K) F_{S_{2,k}}(d_2) + F_{S_{1,k}+S_{2,k}}(d_1, d_2, K). \tag{4.4}
\]

Proposition 4.8 Let \( K_i, i = 1, 2 \) be the capital required for each stop loss reinsurance portfolio of the reinsurer such that \( K = K_1 + K_2 \). Given that the reinsurer is in default, if \( (X_1, \ldots, X_{2k}) \sim SME(\beta, Q) \) with \( \beta_{2k} \geq \beta_i, i = 1, \ldots, 2k - 1, d_j > 0, j = 1, 2 \) and \( T_j, j = 1, 2 \) has finite mean, then

\[
U(K_1, K) = \xi_1 \left( F_{S_{2,k}}(d_2) U_{S_{1,k}}(K, d_1, Z(\beta_{2k})) + U_{S_{1,k}}(K, d_1, d_2, Z(\beta_{2k})) \right) \\
+ \sum_{l=1}^{2k} \sum_{j=1}^{2k} \xi_{j_1,j_2,...,j_l} \prod_{m=1}^{l} \gamma_{jm} \left( F_{S_{2,k}}(d_2) U_{S_{1,k}}(K, d_1, Z(\beta_{2k})) \right) \\
+ U_{S_{1,k}}(K, d_1, d_2, Z(\beta_{2k})) - K_1 \mathcal{F}_{R_2}(K).
\]

Example 4.9 Consider the required capital for the entire portfolio \( K \) as TVaR\(_R\)\(_2\)(\( p \)) and for the individual risk \( K_i \) as TVaR\(_R\)\(_p\)\((T_i, R_2)\), \( i = 1, 2 \) presented in Table 4.3. From Table 4.4 we can see that an increase of the confidence level \( p \) yields an uprise of the capital required for the entire portfolio of the reinsurer which in turn decreases the default probability and also the value of the unpaid losses of each portfolio of the ceding insurer.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( K )</th>
<th>( \phi(K) )</th>
<th>( U(K) )</th>
<th>( K_1 )</th>
<th>( U(K_1, K) )</th>
<th>( K_2 )</th>
<th>( U(K_2, K) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>95.00 %</td>
<td>30.10</td>
<td>0.01860</td>
<td>0.19288</td>
<td>19.69</td>
<td>0.15436</td>
<td>10.41</td>
<td>0.03852</td>
</tr>
<tr>
<td>97.50 %</td>
<td>37.40</td>
<td>0.00929</td>
<td>0.09483</td>
<td>25.47</td>
<td>0.07929</td>
<td>11.93</td>
<td>0.01555</td>
</tr>
<tr>
<td>99.00 %</td>
<td>46.85</td>
<td>0.00370</td>
<td>0.03725</td>
<td>33.35</td>
<td>0.03228</td>
<td>13.50</td>
<td>0.00497</td>
</tr>
<tr>
<td>99.90 %</td>
<td>69.92</td>
<td>0.00036</td>
<td>0.00360</td>
<td>53.59</td>
<td>0.00321</td>
<td>16.33</td>
<td>0.00039</td>
</tr>
<tr>
<td>Independence case</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95.00 %</td>
<td>30.41</td>
<td>0.01863</td>
<td>0.19338</td>
<td>20.15</td>
<td>0.15586</td>
<td>10.26</td>
<td>0.03752</td>
</tr>
<tr>
<td>97.50 %</td>
<td>37.73</td>
<td>0.00930</td>
<td>0.09750</td>
<td>25.99</td>
<td>0.07929</td>
<td>11.73</td>
<td>0.01525</td>
</tr>
<tr>
<td>99.00 %</td>
<td>47.21</td>
<td>0.00371</td>
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<td>33.94</td>
<td>0.03237</td>
<td>13.27</td>
<td>0.00494</td>
</tr>
<tr>
<td>99.90 %</td>
<td>70.31</td>
<td>0.00037</td>
<td>0.00360</td>
<td>54.20</td>
<td>0.00320</td>
<td>16.11</td>
<td>0.00040</td>
</tr>
<tr>
<td>Laplace case</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95.00 %</td>
<td>33.14</td>
<td>0.01870</td>
<td>0.19924</td>
<td>22.11</td>
<td>0.16237</td>
<td>11.02</td>
<td>0.03687</td>
</tr>
<tr>
<td>97.50 %</td>
<td>40.68</td>
<td>0.00932</td>
<td>0.09740</td>
<td>28.21</td>
<td>0.08212</td>
<td>12.47</td>
<td>0.01528</td>
</tr>
<tr>
<td>99.00 %</td>
<td>50.40</td>
<td>0.00372</td>
<td>0.03805</td>
<td>36.39</td>
<td>0.03286</td>
<td>14.01</td>
<td>0.00519</td>
</tr>
<tr>
<td>99.90 %</td>
<td>73.89</td>
<td>0.00037</td>
<td>0.00364</td>
<td>56.80</td>
<td>0.00317</td>
<td>17.10</td>
<td>0.00047</td>
</tr>
<tr>
<td>FGM case</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: Default probability, default value option of the reinsurer and unpaid losses of the insurer.
5 Proofs

Proof of Proposition 3.1 The joint tail probability of \((S_{1,k}, S_{2,k})\) is determined in terms of the joint density of \((X_1, \ldots, X_{2k})\) as follows

\[
P(S_{1,k} > u_1, S_{2,k} > u_2) = \int \cdots \int_{x_{1,k} > u_1, x_{2,k} > u_2} h(x) dx_1 \cdots dx_{2k}.
\] (5.1)

Referring to (2.1), the joint density of \((X_1, \ldots, X_{2k})\) is given by (set \(\gamma_{j_{h}} := \mathbb{E}(g_{j_{h}}(X_{j_{h}}))\))

\[
h(x) = \prod_{i=1}^{2k} f_i(x_i) \left(1 + \sum_{h=2}^{2k} \sum_{1 \leq j_{1}, j_{2}, \ldots, j_{h} \leq j_{h}} \prod_{i \in C \setminus \{j_{h}\}} g_{j_{i}}(x_{j_{i}}) - \gamma_{j_{h}}\right)
\]

\[
= \xi_1 \prod_{i=1}^{2k} f_i(x_i) + \xi_{j_1,j_2} \sum_{i \in C \setminus \{j_1,j_2\}} f_i(x_i) + \xi_{j_1,j_2,j_3} \sum_{i \in C \setminus \{j_1,j_2,j_3\}} f_i(x_i) + \ldots + \xi_{j_1,\ldots,j_{2k-1}} \prod_{i=1}^{2k-1} \tilde{f}_i(x_i),
\]

where \(\tilde{f}_i(x_i) = g(x_i)f_i(x_i)\),

\[
\xi_1 = 1 + \sum_{j_1,j_2} \alpha_{j_1,j_2} \gamma_{j_1,j_2} - \sum_{j_1,j_2,j_3} \alpha_{j_1,j_2,j_3} \gamma_{j_1,j_2,j_3} + \ldots + (-1)^{2k} \alpha_{1,\ldots,2k} \prod_{i=1}^{2k} \gamma_i,
\]

\[
\xi_{j_1,j_2} = -\sum_{j_2} \alpha_{j_1,j_2} \gamma_{j_1} + \sum_{j_2,j_3} \alpha_{j_1,j_2,j_3} \gamma_{j_1,j_2,j_3} + \ldots + (-1)^{2k+1} \alpha_{1,\ldots,2k} \prod_{i \in C \setminus \{j_1,j_2\}} \gamma_i,
\]

\[
\xi_{j_1,j_2,j_3} = -\sum_{j_3} \alpha_{j_1,j_2,j_3} \gamma_{j_1,j_2,j_3} + \sum_{j_3,j_4} \alpha_{j_1,j_2,j_3,j_4} \gamma_{j_1,j_2,j_3,j_4} + \ldots + (-1)^{2k} \alpha_{1,\ldots,2k} \prod_{i \in C \setminus \{j_1,j_2,j_3\}} \gamma_i,
\]

where \(C = \{1, \ldots, 2k\}, j_1 \in C \setminus \{j_1\}, j_2 \in C \setminus \{j_1,j_2\}, \ldots, j_{2k} \in C \setminus \{j_1,\ldots, j_{2k-1}\}\).

After some rearrangements, one can express \(h(x)\) as follows

\[
h(x) = \xi_1 \prod_{i=1}^{2k} f_i(x_i) + \sum_{l=1}^{2k} \sum_{j_1,j_2,\ldots,j_l} \xi_{j_1,j_2,\ldots,j_l} \prod_{h=1}^{l} \tilde{f}_{j_h}(x_{j_h}) \prod_{i \notin \{j_1,j_2,\ldots,j_l\}} f_i(x_i)
\]

\[
= \xi_1 \prod_{i=1}^{2k} f_i(x_i) + \sum_{l=2}^{2k} \sum_{j_1,j_2,\ldots,j_l} \xi_{j_1,j_2,\ldots,j_l} \prod_{i=1}^{l} f^*_i(x_i),
\]

where for \(i = 1, \ldots, 2k\)

\[
f^*_i(x_i) = \begin{cases} f_i(x_i) & \text{if } i \notin \{j_1,j_2,\ldots,j_l\}, \\ \tilde{f}_i(x_i) & \text{if } i \in \{j_1,j_2,\ldots,j_l\}. \end{cases}
\]

By Lemma A.4, \(\tilde{f}_i\) is a pdf of a mixed Erlang distribution, therefore one can write (5.1) as a sum product of convolutions of mixed Erlang risks as follows

\[
P(S_{1,k} > u_1, S_{2,k} > u_2)
= \xi_1 \int_{u_1}^{\infty} \cdots \int_{u_1-x_1}^{\infty} \cdots \int_{u_1-x_{k-1}}^{\infty} \prod_{i=1}^{k-1} f_i(x_i) F_k(u_1 - x_1 - \ldots - x_{k-1}) dx_{k-1} \cdots dx_1
\]
Provided that $\beta_2 \geq \beta, i = 1, \ldots, 2k - 1$, by Lemma A.5 each $i$–th mixed Erlang component of (5.2) can be transformed into a new mixed Erlang distribution with a common scale parameter $Z(\beta_{2k})$. Therefore, with the help of Remark A.7, by convolution (5.2) can be expressed as a sum product of two mixed Erlang survival function as follows

$$\mathbb{P}(S_{1,k} > u_1, S_{2,k} > u_2) = \xi_1 F_{S_{1,k},S_{2,k}}(u_1) F_{S_{2,k}}(u_2) + \sum_{l=1}^{2k} \sum_{j_1,j_2,\ldots,j_l} \xi_{j_1,j_2,\ldots,j_l} \prod_{i=1}^{l} \gamma_{j_i} F_{S_{1,k}}(u_1) F_{S_{2,k}}(u_2).$$

Thus the proof is complete. 

**Proof of Proposition 4.1** Similarly to the independence case described in Lemma A.2, the df of $R_2$ is of mixed distribution and can be expressed in terms of the joint df of $(T_1, T_2)$ as follows

$$F_{R_2}(s) = \begin{cases} \mathbb{P}(T_1 = 0, T_2 = 0) & \text{for } s = 0 \\ \mathbb{P}(T_1 = 0, 0 < T_2 \leq s) + \mathbb{P}(0 < T_1 \leq s, T_2 = 0) + \mathbb{P}(T_1 + T_2 \leq s, 0 < T_1 \leq s, 0 < T_2 \leq s) & \text{for } s > 0 \end{cases}$$

$$= \left\{ \begin{array}{ll} F_{S_{1,k},S_{2,k}}(d_1, d_2) & \text{for } s = 0 \\ F_{S_{1,k},S_{2,k}}(d_1, d_2) + F_{S_{1,k},S_{2,k}}(d_1 + s, d_2) + F_{S_{1,k},S_{2,k}}(s + d_1, d_2) & \text{for } s > 0 \end{array} \right.$$

By Proposition 3.1 and Lemma A.2, $F_{R_2}(s)$ can be written in two terms as follows:

- the discrete term
  $$F_{S_{1,k},S_{2,k}}(d_1, d_2) = \xi_1 F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2) + \sum_{l=1}^{2k} \sum_{j_1,j_2,\ldots,j_l} \xi_{j_1,j_2,\ldots,j_l} \prod_{i=1}^{l} \gamma_{j_i} F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2),$$

- the continuous term
  $$F_{S_{1,k},S_{2,k}}(d_1 + s, d_2 + s) = \xi_1 F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2 + s) + \sum_{l=1}^{2k} \sum_{j_1,j_2,\ldots,j_l} \xi_{j_1,j_2,\ldots,j_l} \prod_{i=1}^{l} \gamma_{j_i} F_{S_{1,k}}(d_1) F_{S_{2,k}}(d_2 + s)$$
  $$+ \xi_1 F_{S_{1,k}}(d_1 + s) F_{S_{2,k}}(d_2) + \sum_{l=1}^{2k} \sum_{j_1,j_2,\ldots,j_l} \xi_{j_1,j_2,\ldots,j_l} \prod_{i=1}^{l} \gamma_{j_i} F_{S_{1,k}}(d_1 + s) F_{S_{2,k}}(d_2)$$
  $$+ \xi_1 F_{S_{1,k}}(d_1, d_2 + s) + \sum_{l=1}^{2k} \sum_{j_1,j_2,\ldots,j_l} \xi_{j_1,j_2,\ldots,j_l} \prod_{i=1}^{l} \gamma_{j_i} F_{S_{1,k}+S_{2,k}}(d_1, d_2 + s).$$

This completes the proof. 

**Proof of Proposition 4.4** In view of (4.3)

$$TVaR_\beta(T_1, T_2) = \frac{\mathbb{E}(T_1 \mathbb{1}_{\{R_2 > VaR_{R_2}(\beta)\}})}{1 - p} = \frac{1}{1 - p} \int_{VaR_{R_2}(\beta)}^{\infty} \mathbb{E}(T_1 \mathbb{1}_{\{R_2 > s\}}) ds.$$  (5.3)
First, we need to calculate $\mathbb{E}(T_1 \mathbb{1}_{(R_2 = s)})$ as follows

$$
\mathbb{E}(T_1 \mathbb{1}_{(R_2 = s)}) = \int_0^\infty u f_{T_1, T_1 + T_2 = s}(u) du.
$$

Let

$$
f_{S_1, k + S_2, k}(d_1, d_2, u) := \frac{d}{du} F_{S_1, k + S_2, k}(d_1, d_2, u),
$$

$$
f_{\bar{S}_1, k + \bar{S}_2, k}(d_1, d_2, u) := \frac{d}{du} F_{\bar{S}_1, k + \bar{S}_2, k}(d_1, d_2, u).
$$

As in Proposition 4.1, one can express $\mathbb{E}(T_1 \mathbb{1}_{(R_2 = s)})$ as follows

$$
\begin{align*}
\mathbb{E}(T_1 \mathbb{1}_{(R_2 = s)}) &= \xi_1 \left( F_{S_2, k}(d_2) \int_0^s u f_{S_1, k}(d_1 + u) du + \int_0^s u f_{S_1, k + S_2, k}(d_1, d_2, u) du \right) \\
&\quad + \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \frac{\xi_{j_1, j_2, \ldots, j_l}}{\prod_{m=1}^l \gamma_m} \left( F_{\bar{S}_2, k}(d_2) \int_0^s u f_{\bar{S}_1, k}(d_1 + u) du \right) \\
&\quad + \int_0^s u f_{\bar{S}_1, k + \bar{S}_2, k}(d_1, d_2, u) du.
\end{align*}
$$

(5.4)

By Lemma A.1, for $X_i \sim ME(\beta, Q_i)$ and $d_i > 0$, $i = 1, 2$

$$
\int_0^s u f_X(d_i + u) du = \frac{1}{\beta} \sum_{k=0}^{\infty} (k + 1) \Delta_k(d_i, \beta, Q_i) \overline{W}_{k+2}(s, \beta) =: \overline{U}_{X_i}(s, d_i, \beta).
$$

(5.5)

Similarly, by Lemma A.2

$$
\int_0^s u f_{X_1 + X_2}(d_1, d_2, u) du = \frac{1}{\beta} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k + 1) \Delta_k(d_1, \beta, Q_1) \Delta_j(d_2, \beta, Q_2) W_{k+j+3}(s, \beta) =: \overline{U}_{X_i}(s, d_1, d_2, \beta).
$$

(5.6)

Taking (5.5) and (5.6) into account, one may write (5.4) as follows

$$
\begin{align*}
\mathbb{E}(T_1 \mathbb{1}_{(R_2 = s)}) &= \xi_1 \left( F_{S_2, k}(d_2) \overline{U}_{S_1, k}(s, d_1, Z(\beta_{2k})) + \overline{U}_{S_1, k + S_2, k}(s, d_1, d_2, Z(\beta_{2k})) \right) \\
&\quad + \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \frac{\xi_{j_1, j_2, \ldots, j_l}}{\prod_{m=1}^l \gamma_m} \left( F_{\bar{S}_2, k}(d_2) \overline{U}_{\bar{S}_1, k}(s, d_1, Z(\beta_{2k})) \right) \\
&\quad + \overline{U}_{\bar{S}_1, k}(s, d_1, d_2, Z(\beta_{2k})).
\end{align*}
$$

(5.7)

Therefore, referring to (5.3) (set $x_p := VaR_p(R_2)$)

$$
TVaR_p(T_1, R_2) = \frac{1}{1 - p} \left[ \xi_1 \left( F_{S_2, k}(d_2) \int_{x_p}^\infty \overline{U}_{S_1, k}(s, d_1, Z(\beta_{2k})) ds + \int_{x_p}^\infty \overline{U}_{S_1, k + S_2, k}(s, d_1, d_2, Z(\beta_{2k})) ds \right) \\
+ \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \frac{\xi_{j_1, j_2, \ldots, j_l}}{\prod_{m=1}^l \gamma_m} \left( F_{\bar{S}_2, k}(d_2) \int_{x_p}^\infty \overline{U}_{\bar{S}_1, k}(s, d_1, Z(\beta_{2k})) ds \right) \\
+ \int_{x_p}^\infty \overline{U}_{\bar{S}_1, k}(s, d_1, d_2, Z(\beta_{2k})) ds \right].
$$

Hence, the result follows easily.

Proof of Proposition 4.6 By definition

$$
U(K) = \mathbb{E} \{ (R_2 - K)_+ \} = \mathbb{P}_{R_2}(K) \mathbb{E} \left\{ R_2 - K \mid R_2 > K \right\}.
$$

Hence in view of Remark 4.2,

$$
U(K) = \xi_1 \left( F_{S_1, k}(d_1) \overline{U}_{S_2, k}(K, d_2, Z(\beta_{2k})) + F_{S_2, k}(d_2) \overline{U}_{S_1, k}(K, d_1, Z(\beta_{2k})) + \overline{U}_{S_1, k + S_2, k}(K, d_1, d_2, Z(\beta_{2k})) \right)
$$
\[ + \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \xi_{j_1, j_2, \ldots, j_l} \prod_{m=1}^{l} \gamma_{j_m} \left( F_{S_{1,k}}(d_1) U_{S_{2,k}}(K, d_2, Z(\beta_{2k})) + F_{S_{2,k}}(d_2) U_{S_{1,k}}(K, d_1, Z(\beta_{2k})) + U_{S_{1,k}}(K, d_1, d_2, Z(\beta_{2k})) \right) \]

establishing the proof.

\[ \square \]

**Proof of Proposition 4.8** The unpaid losses of the ceding insurer line of business is defined as follows

\[ U(K_1, K) = \mathbb{E} \{(T_1 - K_1) 1_{\{R_2 > K\}}\} = \int_{K}^{\infty} \mathbb{E}(T_1 1_{\{R_2 = s\}}) ds - K_1 F_{R_2}(K). \]

In light of (5.7)

\[ U(K_1, K) = \xi_1 \left( F_{S_{2,k}}(d_2) U_{S_{1,k}}(K, d_1, Z(\beta_{2k})) + U_{S_{1,k}}(K, d_1, d_2, Z(\beta_{2k})) \right) \]

\[ + \sum_{l=1}^{2k} \sum_{j_1, j_2, \ldots, j_l} \xi_{j_1, j_2, \ldots, j_l} \prod_{m=1}^{l} \gamma_{j_m} \left( F_{S_{2,k}}(d_2) U_{S_{1,k}}(K, d_1, Z(\beta_{2k})) + U_{S_{1,k}}(K, d_1, d_2, Z(\beta_{2k})) \right) - K_1 F_{R_2}(K). \]

Hence the proof is complete.

\[ \square \]

**Appendices**

**Appendix A  Properties of mixed Erlang distribution**

**Lemma A.1** For a deductible \( d > 0 \), if \( X \sim ME(\beta, Q) \) then the df of \( Y := (X - d)_+ \) is given by

\[ F_Y(y) = \begin{cases} F_X(d) & \text{for } y = 0, \\ F_X(y + d) & \text{for } y > 0, \end{cases} \tag{A.1} \]

where

\[ F_X(y + d) = \sum_{k=0}^{\infty} \Delta_k(d, \beta, Q) W_{k+1}(y, \beta), \]

with

\[ \Delta_k(d, \beta, Q) = \frac{1}{\beta} \sum_{j=0}^{\infty} q_{j+k+1} w_{j+1}(d, \beta). \tag{A.2} \]

**Lemma A.2** Let \( X_1 \) and \( X_2 \) be two independent risks such that \( X_i \sim ME(\beta_i, Q_i), i = 1, 2 \). If \( d_i, i = 1, 2 \) are positive then \( R_2 = Y_1 + Y_2 \), with \( Y_i = (X_i - d_i)_+, i = 1, 2 \), has df

\[ F_{R_2}(s) = \begin{cases} F_{X_1}(d_1) F_{X_2}(d_2) & \text{for } s = 0, \\ F_{X_1}(d_1) F_{X_2}(s + d_2) + F_{X_2}(d_2) F_{X_1}(s + d_1) + F_{X_1 + X_2}(d_1, d_2, s) & \text{for } s > 0, \end{cases} \tag{A.3} \]

where

\[ F_{X_1 + X_2}(d_1, d_2, s) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \Delta_k(d_1, \beta, Q_1) \Delta_j(d_2, \beta, Q_2) W_{k+j+2}(s, \beta). \]
Remarks A.3 Given the tractable of the df of $R_2$, the VaR of $R_2$ at a confidence level of $p \in (0, 1)$ is the solution of

$$F_X(x_1)F_X(x_2) + F_X(x_1)F_X(x_2)\text{VaR}_{R_2}(p) + d_2 + F_X(x_2)F_X(x_2)\text{VaR}_{R_2}(p) + d_1 + F_T(\text{VaR}_{R_2}(p)) = p,$$

which can be solved numerically. In addition, the TVaR of $R_2$ at a confidence level $p \in (0, 1)$ is given by (set $x_p := \text{VaR}_{R_2}(p)$)

$$TVaR_{R_2}(p) = \frac{1}{1 - p} \left( F_X(x_1)U_X(x_p, d_1, \beta) + F_X(x_2)U_X(x_p, d_2, \beta) + U_{X_1 + X_2}(x_p, d_1, d_2, \beta) \right),$$

where

$$U_X(x_p, d, \beta) = \sum_{k=0}^\infty (k + 1)\Delta_k(d, \beta, Q)d_{k+2}(x_p, \beta),$$

$$U_{X_1 + X_2}(x_p, d_1, d_2, \beta) = \sum_{k=0}^\infty \sum_{j=0}^\infty (k + j + 2)\Delta_k(d_1, \beta, Q_1)\Delta_j(d_2, \beta, Q_2)d_{k+j+3}(x_p, \beta).$$

Proof. Since $X_1$ and $X_2$ are independent risks which have mixed distribution, the df of $R_2$ can also be expressed as a df of a mixed distribution which depends on the value of $s$ as follows:

- the discrete part of $F_{R_2}$ is obtained for $s = 0$, specifically we have

$$F_{R_2}(0) = P(Y_1 + Y_2 \leq 0) = P(Y_1 + Y_2 = 0) = P(Y_1 = 0, Y_2 = 0) = F_X(d_1)F_X(d_2), \quad (A.4)$$

- for $s > 0$ the continuous part of $F_{R_2}$ is given by

$$F_{R_2}(s) = P(Y_1 + Y_2 \leq s)$$

$$= \sum_{k=0}^\infty \sum_{j=0}^\infty \Delta_k(d_1, \beta_1)\Delta_j(d_2, \beta_2)\int_0^s W_{k+1}(s, \beta)w_{j+1}(u, \beta)du. \quad (A.6)$$

Let $F_T(s) := \int_0^s F_X(s - u + d_1)f_X(u + d_2)du$, by Lemma A.1 this can be written as

$$F_T(s) = \sum_{k=0}^\infty \sum_{j=0}^\infty \Delta_k(d_1, \beta_1)\Delta_j(d_2, \beta_2)\int_0^s W_{k+1}(s, \beta)w_{j+1}(u, \beta)du. \quad (A.6)$$

It can be seen that $\int_0^s W_{k+1}(s, \beta)w_{j+1}(u, \beta)du$ is a convolution of two independent Erlang risks with a common scale parameter $\beta$, which is again an Erlang risk with shape parameter $k + j + 2$ and scale parameter $\beta$. Thus combining (A.4), (A.5) and (A.6) the claim follows easily.

Lemma A.4 Let $X \sim ME(\beta, Q)$ with pdf $f(x, \beta, Q)$, if $g$ is some positive function such that $E\{g(X)\} < \infty$, then $c(x, \beta, Q) = \frac{g(x)f(x, \beta, Q)}{E\{g(X)\}}$ is again a pdf of mixed Erlang distribution with scale parameter $Z(\beta)$ and mixing weights $\Theta(Q) = (\theta_1, \theta_2, \ldots)$, with

$$c(x, \beta, Q) = \sum_{k=1}^\infty \theta_k w_k(x, Z(\beta)),$$

where

- $Z(\beta) = 2\beta$ and $\theta_k = \frac{1}{2^{k-1}} \sum_{j=1}^k \binom{k-1}{j-1} q_j \sum_{i=k-j+1}^\infty q_i$, for $g(x) = 2\mathbb{P}(x), \quad (A.6)$
\* Z(\beta) = \beta and
\[
\theta_k = \begin{cases} 
0 & \text{for } k \leq t, \\
\frac{q_{k-1} \Gamma(k)}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+1)}{\Gamma(j)}} & \text{for } k > t,
\end{cases}
\]
for \( g(x) = x^t \) with \( t \in \mathbb{R} \).

\* Z(\beta) = \beta + t and \( \theta_k = \frac{q_k \beta^k}{\sum_{j=1}^{\infty} q_j \beta^j} \) with \( \beta = \frac{\beta}{\beta + t} \), for \( g(x) = e^{-tx} \) with \( t \in \mathbb{N} \).

**Proof.** We have
\[
c(x, \beta, Q) = \frac{g(x)f(x, \beta, Q)}{\mathbb{E}\{g(X)\}} = \frac{1}{\mathbb{E}\{g(X)\}} \sum_{k=1}^{\infty} q_k \frac{\beta^k}{(k-1)!} g(x)x^{k-1}e^{-\beta x}.
\]
For \( g(x) = x^t \) one can write (A.7) as follows
\[
c(x, \beta, Q) = \frac{1}{\mathbb{E}\{X^t\}} \sum_{k=1}^{\infty} q_k \frac{\beta^k}{(k-1)!} x^{t+k-1}e^{-\beta x}
\]
\[
= \sum_{k=1}^{\infty} \left( \frac{q_k \Gamma(k+t)}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+t)}{\Gamma(j)}} \right) w_{k+t}(x, \beta)
\]
\[
= \sum_{s=t+1}^{\infty} \left( \frac{q_s - t \Gamma(1-t)}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+1)}{\Gamma(j)}} \right) w_s(x, \beta)
\]
\[
= \sum_{s=1}^{\infty} \theta_s w_s(x, \beta),
\]
with
\[
\theta_s = \begin{cases} 
0 & \text{for } s \leq t, \\
\frac{q_{s-1} \Gamma(s)}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+s-1)}{\Gamma(j)}} & \text{for } s > t,
\end{cases}
\]
For \( g(x) = e^{-tx} \), (A.7) can be expressed as follows (set \( \bar{\beta} := \frac{\beta}{\beta + t} \))
\[
c(x, \beta, Q) = \frac{1}{\mathbb{E}\{e^{-tx}\}} \sum_{k=1}^{\infty} q_k \frac{\beta^k}{(k-1)!} x^{k-1}e^{-(\beta+t)x}
\]
\[
= \sum_{k=1}^{\infty} \left( \frac{q_k \beta^k}{\sum_{j=1}^{\infty} q_j \bar{\beta}^j} \right) w_k(x, \beta + t)
\]
\[
= \sum_{k=1}^{\infty} \theta_k w_k(x, \beta + t).
\]
For \( g(x) = 2F(x) \), see [2] for the proof.

The results presented in the next two lemmas can be found in Section 2.2 of [24] and Section 7.2 of [11], respectively.

**Lemma A.5** If \( X \sim ME(\beta_1, Q) \), then for any positive constant \( \beta_2 \geq \beta_1 \) we have
\[
X \sim ME(\beta_2, \Psi(Q)), \quad \Psi(Q) = (\psi_1, \psi_2, \ldots),
\]
where
\[
\psi_k = \sum_{i=1}^{k} q_i \binom{k-1}{i-1} \left( \frac{\beta_1}{\beta_2} \right)^i \left( 1 - \frac{\beta_1}{\beta_2} \right)^{k-i}, \quad k \geq 1.
\]
Lemma A.6 Let $X_1, X_2$ be two independent random variables such that $X_i \sim ME(\beta, Q_i), i = 1, 2$, then $S_2 = X_1 + X_2 \sim ME(\beta, \Pi\{Q_1, Q_2\})$ with

$$\pi_l\{Q_1, Q_2\} = \begin{cases} 0 & \text{for } l = 1, \\ \sum_{j=1}^{l-1} q_{1,j} q_{2,l-j} & \text{for } l > 1. \end{cases}$$

Remarks A.7 According to Cossette et al. (2012) (Remark 2.1), the results in Lemma A.6 can be extended to $S_n = \sum_{i=1}^{n} X_i$, provided that $X_1, \ldots, X_n$ are independent, $X_i \sim ME(\beta, Q_i)$ for $i = 1, \ldots, n$. Specifically, $S_n \sim ME(\beta, \Pi\{Q_1, \ldots, Q_n\})$ where the individual mixing probabilities can be evaluated iteratively as follows

$$\pi_l\{Q_1, \ldots, Q_{n+1}\} = \begin{cases} 0 & \text{for } l = 1, \ldots, n, \\ \sum_{j=n}^{l-1} \pi_j\{Q_1, \ldots, Q_n\} q_{n+1,l-j} & \text{for } l = n + 1, n + 2, \ldots. \end{cases}$$

Appendix B Joint density of sums of Sarmanov random vectors

One of the main features of the Sarmanov distribution is that its pdf can be used to derive some results in an analytical way. For instance Vernic [22] have derived general formula for the density of the sum of several rv joined by the Sarmanov distribution. Below we derive the joint density of $n$ random vectors where each vector consists of $k$ elements and we denote the sum of elements within each random vector as $S_i := \sum_{j=(i-1)k+1}^{ik} X_j, i = 1, \ldots, n$. Furthermore, we assume that the joint distribution of the overall random vectors $(X_1, \ldots, X_{nk})$ has the Sarmanov distribution with any kernel function satisfying (2.2).

Theorem B.1 The joint density of $(S_1, \ldots, S_n)$ is given by

$$\zeta(u_1, \ldots, u_n) = \prod_{i=1}^{n} f_{S_i}(u_i) + \sum_{h=2}^{n} \sum_{j_1 < j_2 < \ldots < j_h \leq n} \alpha_{j_1, \ldots, j_h} \prod_{i=1}^{n} f^{(*)}_{S_i}(u_i),$$

where

$$f_{S_i}(u_i) = (f_{(i-1)k+1} \ast \ldots \ast f_{ik})(u_i),$$

$$f^{(*)}_{S_i}(u_i) = (f^{(*)}_{(i-1)k+1} \ast \ldots \ast f^{(*)}_{ik})(u_i),$$

with

$$f^{(*)}_m(x_m) = \begin{cases} f_m(x_m) & \text{if } m \notin \{j_1, j_2, \ldots, j_h\}, \\ \phi_{x_m}(s)f_m(x_m)ds & \text{if } m \in \{j_1, j_2, \ldots, j_h\}, m = 1, \ldots, nk. \end{cases}$$

Proof. The joint density of $(S_1, \ldots, S_n)$ is determined in term of the joint density of $(X_1, \ldots, X_{nk})$ as follows

$$\zeta(u_1, \ldots, u_n) = \int \ldots \int_{s_1 = u_1, s_2 = u_2, \ldots, s_n = u_n} h(x)dx_1 \ldots dx_{nk-1}, \quad (B.1)$$

with $x = (x_1, \ldots, x_{nk}), s_1 = x_1 + \ldots + x_k, s_2 = x_{k+1} + \ldots + X_{2k}, s_n = x_{nk-k+1} + \ldots + x_{nk}$.

Refering to (2.1),

$$h(x) = \prod_{i=1}^{nk} f_i(x_i) \left( 1 + \sum_{h=2}^{nk} \sum_{j_1 < j_2 < \ldots < j_h \leq nk} \alpha_{j_1, \ldots, j_h} \prod_{k=1}^{h} \phi_{j_k}(x_{j_k}) \right)$$

$$= \prod_{i=1}^{nk} f_i(x_i) + \sum_{h=2}^{nk} \sum_{j_1 < j_2 < \ldots < j_h \leq nk} \alpha_{j_1, \ldots, j_h} \prod_{k=1}^{h} \phi_{j_k}(x_{j_k}) f_j(x_j) \prod_{m \notin \{j_1, j_2, \ldots, j_h\}} f_m(x_m)$$

$$= \prod_{i=1}^{nk} f_i(x_i) + \sum_{h=2}^{nk} \sum_{j_1 < j_2 < \ldots < j_h \leq nk} \alpha_{j_1, \ldots, j_h} \prod_{m=1}^{nk} f^{(*)}_m(x_m).$$
where for \( m = 1, \ldots, nk \)

\[
f_m^*(x_m) = \begin{cases} 
  f_m(x_m) & \text{if } m \notin \{j_1, j_2, \ldots, j_h\}, \\
  \phi_m(x_m) f_m(x_m) & \text{if } m \in \{j_1, j_2, \ldots, j_h\}.
\end{cases}
\]

Therefore, one can express (B.1) as a sum of convolutions as follows (set \( x_{ik} := (x_{(i-1)k+1}, \ldots, x_{ik-1}) \), \( i = 1, \ldots, n \))

\[
\zeta(u_1, \ldots, u_n) = \prod_{i=1}^{n} \int_{R^{k-1}} \prod_{m=(i-1)k+1}^{ik-1} f_j(x_j) f_{ik}(u_i - \sum_{m=(i-1)k+1}^{ik-1} x_m) \, dx_{ik} \\
+ \sum_{h=2}^{n} \sum_{1 \leq j_1 < j_2 < \ldots < j_h \leq n} \alpha_{j_1, \ldots, j_h} \prod_{i=1}^{n} \int_{R^{k-1}} \prod_{m=(i-1)k+1}^{ik-1} f_m^*(x_m) f_{ik}^*(u_j - \sum_{m=(i-1)k+1}^{ik-1} x_m) \, dx_{ik} \\
= \prod_{i=1}^{n} f_{S_i}(u_i) + \sum_{h=2}^{n} \sum_{1 \leq j_1 < j_2 < \ldots < j_h \leq n} \alpha_{j_1, \ldots, j_h} \prod_{i=1}^{n} f_{S_i}^*(u_i).
\]

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References


