Institute of Actuaries of Australia

Fair Value of Liabilities
-
How Do We Define "Closest" Asset Match?

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Fair Value of Liabilities - How Do We Define "Closest" Asset Match?

Abstract

"Fair value" of liabilities is becoming an increasingly important concept for the accounts of insurance companies. The fair value of assets can, in most cases, be determined by taking the market value but there is no practical market value for insurance liabilities. This leads to various alternative approaches. While the theory is still under active discussion, all approaches use the concept of the asset portfolio which is the "closest match" to the liability portfolio. The discount rate implied by that asset portfolio is used as the discount rate for the liability cashflows to give a "fair value" of liabilities for the insurer's balance sheet.

In this paper we suggest an approach to find the asset portfolio with the "closest match" to a particular liability portfolio when both are comprised of stochastic cashflows. Since these portfolios are stochastic the approach results in "closest match" being measured on a probabilistic basis.

Our definition of "closest match" is

> The asset portfolio which, for a given probability of ultimate surplus being negative, requires the lowest initial asset amount.

This definition leads also to the conclusion that the "closest match" asset portfolio can be different for different probabilities of insolvency.

A worked example using stochastic models for both asset and liability cashflows shows the "closest match" portfolios for various probabilities of insolvency.

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1. Introduction

“Fair value” of liabilities is becoming an increasingly important concept for the accounts of insurance companies. The fair value of assets can, in most cases, be determined by taking the market value but there is no practical market value for insurance liabilities. This leads to various alternative approaches. While the theory is still under active discussion all approaches use the concept of the asset portfolio which is the “closest match” to the liability portfolio. The discount rate implied by that asset portfolio is used as the discount rate for the liability cashflows to give a “fair value” of liabilities for the insurer’s balance sheet.

When a future insurance liability is known with certainty, (for example, it is known that exactly $4000 liability is due in 2 years’ time), the fair value will be very straightforward to determine, i.e. the value of 2-year zero coupon bond. However, few general insurance liabilities are known with certainty. In particular, for long tail business there are very large uncertainties surrounding the liability cashflows. Therefore, a perfect match usually does not exist for most general insurance liabilities and the idea of identifying the asset portfolio with the “closest match” forms an important component of determining the fair value of liabilities.

This particular scenario is representative of a more general problem which is relevant not only to the fair value of liabilities but to the overall concept of asset / liability matching, i.e. how do we determine the asset portfolio with the closest match to a liability portfolio when both are comprised of stochastic cashflows.

In this paper we suggest an approach to find the asset portfolio with the “closest match” to a particular liability portfolio when both are comprised of stochastic cashflows. Since these portfolios are stochastic the approach results in “closest match” being measured on a probabilistic basis.

The paper is structured as follows –

Introduction
    This introduction
2 Definition of “Closest Match”
    The derivation of a definition of “closest match”
3 A Worked Example
  3.1 Liability Data
    Description of the liabilities modelled
  3.2 Stochastic liability model
    The particular stochastic liability model used

2 Definition of "closest match"

Wise (1984) defines a "positive match asset portfolio" to a given liability as the one which will minimise the mean square ultimate surplus. Ultimate surplus is measured in terms of the realisable market value of the assets remaining when all liabilities have been extinguished. The term 'positive' refers to the exclusion of negative holding of assets, namely borrowings. Wise also shows how a "positive match asset portfolio" can be obtained for a fixed liability known with certainty.

Wilkie (1985) points out that the positive matching portfolio identified by Wise (1984) might not be an efficient portfolio, and even if it is efficient under some circumstances, it might not be the most optimal portfolio for a particular investor.

Wilkie (1985) argues that rational investors must take account of the prices of securities in order to choose an optimal portfolio. Therefore, Wilkie considers feasible portfolios in the P-E-V 3-dimensional space, where P represents the aggregate price of all assets in the portfolio, E the expected ultimate surplus of assets net of liabilities on completion of the liability cash flows and V the variance of ultimate surplus. Wilkie has therefore generalized conventional portfolio theory by including the price P of the portfolio as a third dimension. In the conventional theory (described by, for example, Moore), only E and V are considered because, in the absence of fixed unmarketable liabilities, the proportions of assets to be held in the selected portfolio will be the same whatever the value of P. In order to identify the efficient portfolio, he assumes that investors are in favour of a high expected surplus, E, low variance of surplus, V, and a low immediate price, P. Wilkie also shows how the particular preferences of an investor can be expressed and used to select particular portfolios from the range of efficient portfolios.

In this paper, unlike Wise or Wilkie, the authors decided to make a stochastic assumption for liabilities which might more closely model the behaviour of general insurers' liabilities.

For most insurers or pension funds, it has long been argued that it may not be the most appropriate way to define risk by variance. For these financial institutions, the major risk concern is the failure to meet future liabilities, which is the insolvency risk.

Overriding any considerations of theoretical asset / liability profiles insurers must ensure that they remain solvent at all times. As a result if the assets become less than the liabilities the "game" is over. Hence any definition of "closest match" must take into account the probability of insolvency. Insurer's business strategies are always based around optimising profit while providing for an acceptably low probability of insolvency.

For the purposes of this discussion insolvency is defined as an ultimate surplus of less than zero. Actual insolvency may have arisen at some earlier time but since we are only examining the position when the liability cashflows have ceased, insolvency is defined at that time. It is clear that, with asset and liability cashflows both being stochastic in nature, any finite initial amount of assets still results in a non-zero probability of insolvency. It is only with an infinite amount of assets and a finite amount of liabilities that the probability of insolvency is zero – hardly a likely real world scenario!

It also follows that for any particular asset portfolio the higher the initial amount of assets the lower the probability of insolvency. In addition, for any given amount of initial assets the probability of insolvency for each possible asset portfolio will be different. These two propositions also lead to the observation that for a given probability of insolvency each possible asset portfolio will require a different initial amount of assets to ensure a probability of insolvency equal to that required.

Since we have two variables – probability of insolvency and initial asset amount – and only one equation, typically no unique solution exists. In order to determine the "closest match" we must set one of those variables in order to reach a unique solution. Since insolvency is the ultimate determinant of an insurance company's fate we determine that it is the probability of insolvency which must be set before the "closest match" asset portfolio can be decided.

Our definition of "closest match" is therefore

> The asset portfolio which, for a given probability of ultimate surplus being negative, requires the lowest initial asset amount.

This definition leads also to the conclusion that the “closest match” asset portfolio can be different for different probabilities of insolvency.

In our view, this definition applies quite generally to asset / liability matching not just to the context of fair values of liabilities.

3 A Worked Example

In order to illustrate the application of our definition of “closest match” we now show a worked example of the approach. A hypothetical portfolio of outstanding long tail claims is measured against various asset portfolios which are assumed to follow one of the well known stochastic asset models. The details of both the stochastic liability model and the asset model are described in the following sections.

3.1 Liability Data

In this paper, we suppose there exists a hypothetical portfolio of long tail outstanding claims which runs off in ten years’ time and where payments of claims are made at the end of each quarter. The hypothetical claims payment experience is shown in Appendix A with the row representing the accident quarter of the claim and the column the number of quarters the report or close of the claim is delayed. The data in the triangle includes both general and superimposed inflation.

The claims cashflow model uses the stochastic chain-ladder method suggested by Renshaw and Verrall (1998) and cashflows are then further adjusted by both the general inflation and super-imposed inflation. General inflation is simulated stochastically by using the asset model and superimposed inflation is assumed to be 8% p.a. constantly.

3.2 Stochastic liability model

Renshaw and Verrall (1998) present a statistical model underlying the chain-ladder technique. They show that the estimates produced by chain-ladder method is equivalent to a generalised linear model (GLM) with a log link function relating to the mean of the responses and a Poisson distribution for error structure, $\mu+\alpha_i+\beta_j$ as linear predictor.

Namely,

Y_{ij} ~Poisson with mean m_{ij}, independently $\forall i, j$
where:

$$\log m_{ij} = \mu + \alpha_i + \beta_j$$

and $\alpha_1 = \beta_1 = 0$

Y_{ij} represents the increment claim amount reported with accident time index i and delay index j.

Renshaw and Verrall (1998) also point out that it is easy to write down a quasi-likelihood, which allows for the variance relationship with the mean to be user specified rather than being fixed according to the distribution function of the error distribution. This will allow the model to be applied to negative incremental claims and the results are always the same as those by the chain-ladder technique when $\sum_{i=1}^{n-j+1} Y_{ij} \geq 0$, where n is the maximum of delay index. They also point out the chain-ladder method will be more appropriate if the run-off triangle consists of claim numbers rather than claim amounts. In the light of this, instead of using Poisson distribution for errors, Gamma, log-normal or inverse-Gaussian distribution may be used for claim size.

In this paper, a gamma distribution is assumed for error distribution, log as the link function, and $\mu + \alpha_i + \beta_j$ as linear predictor.

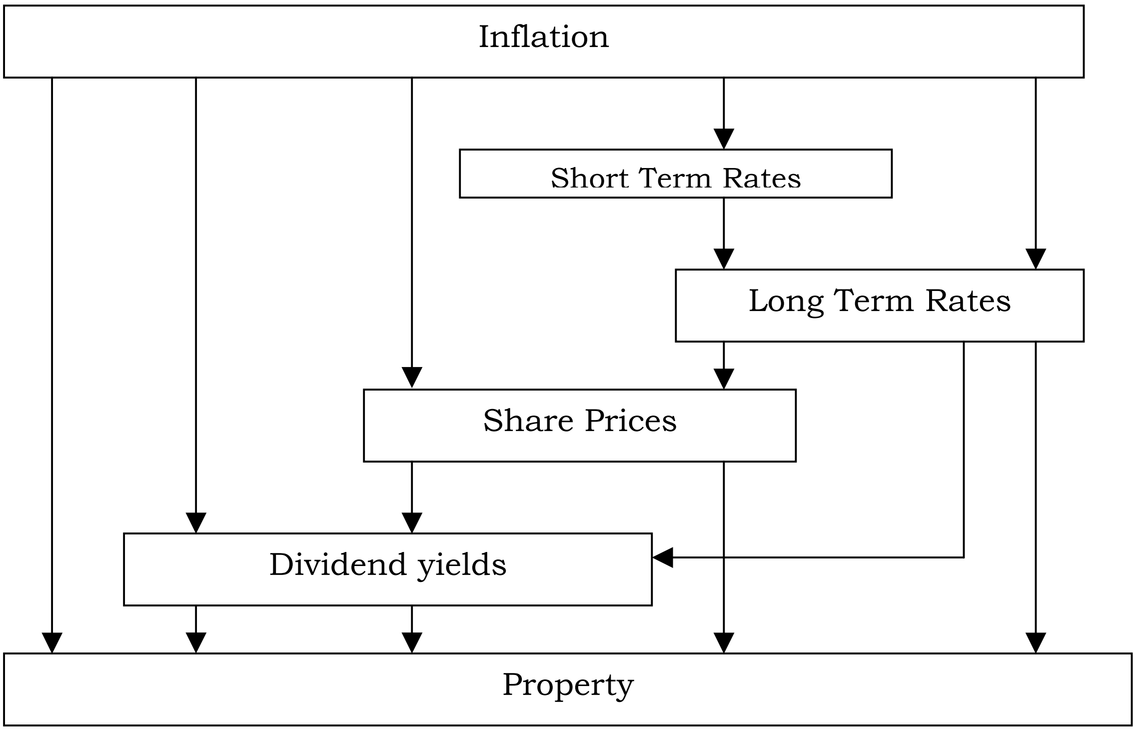
3.3 Stochastic asset model

In this paper the asset classes are restricted to a short-term interest security represented by 90-day bank bills, long-term interest security represented by 10 year Australian government bonds, and equity represented by the Australian All Ordinary Index. For simplicity, in the rest of the paper, the three asset classes will be referred to as cash, bond and equity, respectively. It is assumed that cashflows occur at the end of each quarter. Assets are sold at the end of each quarter to pay off the claims that arise during the quarter. After the sale of assets, the weighting of each asset class is assumed to remain the same through the whole ten years. Transaction costs and taxes are ignored in this paper for simplicity.

3.3.1 Australian Stochastic model

In this worked example, Jon Carter's Australian stochastic model (1991) is used to model the returns for each asset class. Jon Carter's Model (JC model) can be viewed as a modification of the Wilkie model relevant to the Australian financial environment. The cascade methodology, which was adopted in the Wilkie model was also applied in the JC Model. However the cascade in this model was expanded beyond the original Wilkie model (1986). It is represented by Figure 3.1.

Figure 3.1 Cascade Structure of the JC Model



Full details of the asset model are set out in Carter's original paper and, for ease of reference, are shown in Appendix B.

3.3.2 Return of asset classes

Cash

Since 90-day bank bills are held until redemption date, the return of this short-term security for any quarter t will equal the short-term yields at time t-1 predicted by the stochastic asset model.

Bond

10-year government bonds are assumed to be held until redemption except for those which are sold at the end of each quarter in order to meet claims. The return of bonds equals $\frac{P_t - P_{t-1} + C_t}{P_{t-1}}$ where P_t represents the price of bonds at time t and C_t the coupon payment during the t^{th} quarter. The half-yearly coupon rate is assumed to be 6% p.a. Since the JC model models the yield of long-term bonds instead of the price of the bond, bond price is calculated based on the yield predicted by the model. Note that the term to maturity of bonds decreases over time. For example, suppose that at the beginning we add 10-year bonds into our asset portfolio, after one quarter the term-to-maturity of these bonds will be 9.75 years. However the JC model only models yield for the short-term security (90 days) and the long-term bond (10- year government bond). In this paper simple linear interpolation is used to obtain the yield corresponding to the term-to-maturity at the end of each quarter. For instance, at the end of the 4th quarter, the term-to-maturity for bonds will reduce to 9 years and suppose that the asset model predicts the short-term yield and long-term yields to be n_4 and l_4, then the yield used to price the bonds at the end of 4th quarter will equal $n_4+\frac{l_4-n_4}{10-0.25}\times(9-0.25)$.

<u>Equity</u>

The return of equity will equal the sum of the price yields and dividend yields predicted by the asset model.

3.3.3 Parameter estimation

The data required for estimation of the parameters of JC model and how they are measured are summarised in table 3.1. The data used for estimation are the quarterly data series starting from the first quarter of 1980 to the last quarter of 2000.

Table 3.1 Data required to estimate the parameters of JC Model

Economic Variables	**Measurement**
Inflation Rate	CPI
Short-term fixed interest rate	90-Days Bank Accepted Bills
Long-term fixed interest rate	10-year Government Bond
Share dividend yield	Difference between return of AOI Accumulated index and AOI price index
Dividend	Dividend Yield times AOI Price Index in previous year
Share price return	Return of AOI Price Index

The parameters of the JC model were estimated using the minimum variance estimation method and results are summarised in Table 3.2

Table 3.2 Estimates of the JC Model Parameters

Parameters	Estimate	Parameters	Estimate
${}^{q}\theta_1$	-0.6094	${}^{l}\omega_1$	0.2265
${}^{q}\theta_2$	0.6243	${}^{l}s$	0.0015
${}^{q}\theta_3$	-0.9996	${}^{\rho}\Phi_0$	0.0200
${}^{q}s$	0.0070	${}^{\rho}\sigma$	0.1000
${}^{n}\omega_1$	0.1530	${}^{y}\Phi_0$	0.0092
${}^{n}\omega_2$	-0.2055	${}^{y}\omega_1$	0.1166
${}^{n}\theta_3$	-0.2253	${}^{y}\theta_2$	0.0933
${}^{n}s$	0.0039	${}^{y}\Phi_3$	0.7128
		${}^{y}s$	0.0029

3.4 Simulation Approach

Since both the asset and liability models adopted in this paper are stochastic, it is intractable to analytically derive the distribution of ultimate surplus for a given asset portfolio. For this reason, a Monte Carlo simulation method is used to approximate the likely distribution of ultimate surplus. While a very wide range of portfolios could be tested in practice for this worked example, we considered only four. The four are 100% cash, 100% bonds, 100% equities and balanced – 30% cash, 30% bonds, 40% equities.

Apart from future asset returns and future claim payments, the value of the ultimate surplus depends on the value of the initial total asset amount.

The distribution of the ultimate surplus for a portfolio with a given initial asset amount is estimated from the results of 10000 simulations. The probability of a negative ultimate surplus is determined from the number of negative results from the 10,000 simulation results. No account is taken of the size of the negative amounts. All insolvencies are assumed to be “fatal”.

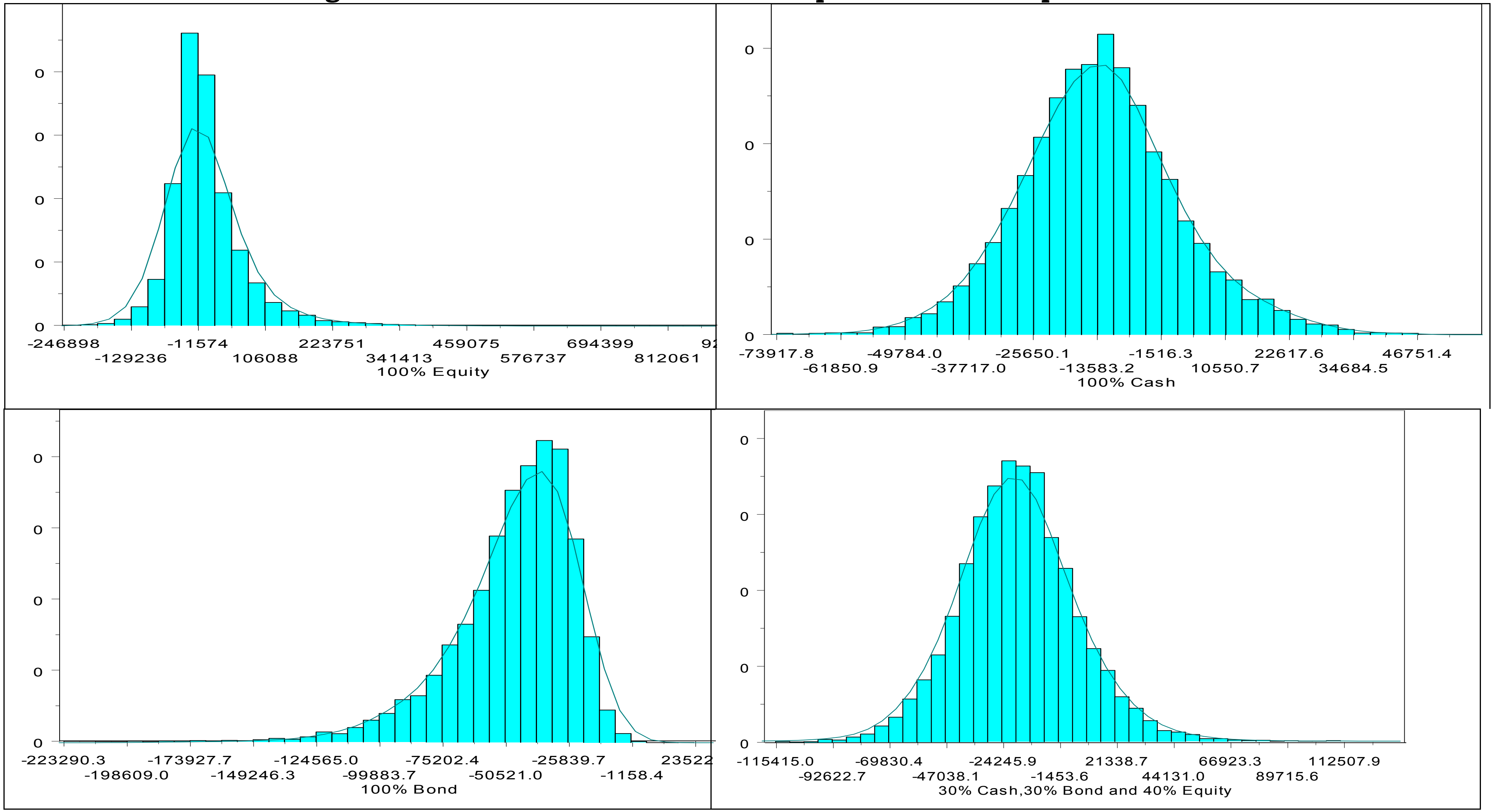
4. Results

4.1 Distribution of ultimate surplus

Figure 4.1 shows the distributions of simulated ultimate surplus resulting from four different asset portfolios with initial asset amount of $75,000. Three of the four asset portfolios contain only one asset class, and the fourth contains 30% cash, 30% bond and 40% equity. It can be seen that the ultimate surplus from all-equity and all-bond asset portfolios exhibits strong skewness. The ultimate surplus from the all-cash portfolio and the balanced portfolio do not exhibit strong skewness. For the balanced portfolio, the positive skewness caused by inclusion of equity might have been counteracted by the negative skewness caused by the inclusion of bonds.

It can be seen from the graphs that the distribution of ultimate surplus may not be normal based on the assumed stochastic asset and liability model. We argue that non–normal ultimate surplus is more likely in practice due to non-normality of either asset return or liability distribution. This non-normality of ultimate surplus implies mere mean and variance of ultimate surplus will not be enough to characterise the behaviour of ultimate surplus.

Figure 4.1 Distributions of ultimate surplus for different portfolios



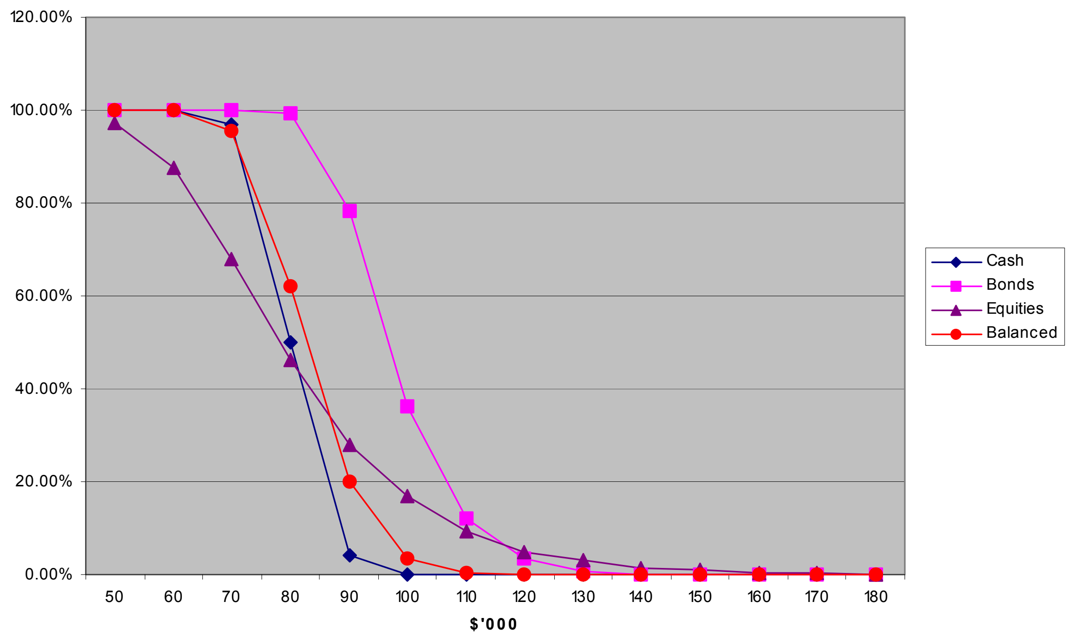
4.2 “Closest Match” Asset Portfolios

The initial asset amounts for given probabilities of insolvency for each of the four illustrative asset portfolios are shown in Table 4.1. Graphs of this same data are shown in Figure 4.2.

Table 4.1 – Probabilities of Insolvency

Initial Asset Amount	All Cash	All Bond	All Equity	Balanced
50000	100.00%	100.00%	97.33%	100.00%
60000	99.99%	100.00%	87.57%	99.96%
70000	96.90%	100.00%	67.78%	95.68%
80000	50.08%	99.37%	46.15%	62.09%
90000	4.27%	78.44%	27.98%	20.16%
100000	0.06%	36.22%	16.99%	3.42%
110000	0.02%	12.06%	9.48%	0.33%
120000	0.03%	3.42%	4.88%	0.02%
130000	0.01%	0.83%	3.02%	0.05%
140000	0.01%	0.10%	1.50%	0.01%
150000	0.04%	0.03%	0.93%	0.03%
160000	0.05%	0.02%	0.43%	0.04%
170000	0.05%	0.04%	0.21%	0.03%
180000	0.01%	0.00%	0.16%	0.01%

Figure 4.2 – Probabilities of Insolvency



For all low probabilities of insolvency the balanced portfolio is the "closest match" with the cash portfolio also very attractive but at much higher probabilities the equity portfolio predominates. This result is likely to follow from the expected volatility in equity returns which, in the balanced portfolio are, as expected, materially reduced by the diversification. In no case does the bonds portfolio suggest itself as a candidate. Clearly the lower expected return is not outweighed by their lower volatility.

It should, however, be noted that these relationships are, to a material extent, dictated by the form of the particular asset model. Using different asset models which reveal just how much is a matter for further investigation.

5. Conclusion

We have defined the "closest match" asset portfolio as

> The asset portfolio which, for a given probability of ultimate surplus being negative, requires the lowest initial asset amount.

We have chosen this definition to recognise that, for most general insurers, it is the risk of insolvency which must take first place in their risk management strategy. Insolvency is almost always "fatal".

Our worked example shows that "closest match" can change dramatically for different probabilities of insolvency. In particular, for very low probabilities - < 1% - the balanced portfolio, or the cash porfolio, is the "closest match", while for high probabilities the equity portfolio is preferred.

We also find that the distribution of ultimate surplus is not normal. This suggests caution when attempting to use shortcuts to extrapolate from known results to other scenarios. Mere mean or variance of ultimate surplus may not be adequate to explain why a particular portfolio has lower or higher probability of insolvency than another.

Further Research

In this paper we illustrate how to obtain a "closest match" asset portfolio using one stochastic asset model and a hypothetical stochastic model of claim cashflows. We plan to continue this research initially by examining the variation in results arising from the use of different asset models. Once this task is completed a complete analysis of the sensitivity of results with different asset models and claims portfolios which cover the range of variation possible e.g. short tail, long tail, annuities, etc. can be prepared. In addition the range of asset portfolios can be suitably extended.

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Appendix A: Claim Run-off Triangle

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	902	301	702	100	586	195	456	65	475	158	369	53	162	54	126	18	312	104	243	35	476	159	370	53	329	110	256	37	108	36	84	12	10	3	8	1	31	10	24	3
2	451	150	351	50	293	98	228	33	237	79	185	26	81	27	63	9	156	52	121	17	238	79	185	26	165	55	128	18	54	18	42	6	5	2	4	1	16	6	13	
3	677	226	526	75	440	147	342	49	356	119	277	40	121	40	94	13	234	78	182	26	357	119	277	40	247	82	192	27	81	27	63	9	7	2	6	1	34	18		
4	226	75	175	25	147	49	114	16	119	40	92	13	40	13	31	4	78	26	61	9	119	40	92	13	82	27	64	9	27	9	21	3	2	1	2	0	17			
5	19	6	15	2	752	251	585	84	200	67	156	22	949	316	738	105	561	187	436	62	327	109	254	36	19	6	14	2	121	40	94	13	96	32	75	11				
6	10	3	7	1	376	125	293	42	100	33	78	11	474	158	369	53	280	93	218	31	164	55	127	18	9	3	7	1	61	20	47	7	50	18	39					
7	14	5	11	2	564	188	439	63	150	50	117	17	711	237	553	79	421	140	327	47	245	82	191	27	14	5	11	2	91	30	71	10	104	56						
8	5	2	4	1	188	63	146	21	50	17	39	6	237	79	184	26	140	47	109	16	82	27	64	9	5	2	4	1	30	10	24	3	54							
9	614	205	477	68	1005	335	781	112	879	293	683	98	408	136	318	45	467	156	363	52	626	209	487	70	117	39	91	13	109	36	84	12								
10	307	102	239	34	502	167	391	56	439	146	342	49	204	68	159	23	233	78	182	26	313	104	244	35	58	19	45	6	56	20	44									
11	460	153	358	51	754	251	586	84	659	220	513	73	306	102	238	34	350	117	272	39	470	157	365	52	88	29	68	10	118	63										
12	153	51	119	17	251	84	195	28	220	73	171	24	102	34	79	11	117	39	91	13	157	52	122	17	29	10	23	3	60											
13	1018	339	792	113	1062	354	826	118	758	253	590	84	990	330	770	110	389	130	302	43	478	159	372	53	177	59	138	20												
14	509	170	396	57	531	177	413	59	379	126	295	42	495	165	385	55	194	65	151	22	239	80	186	27	92	33	72													
15	763	254	594	85	797	266	620	89	568	189	442	63	743	248	578	83	291	97	227	32	359	120	279	40	192	103														
16	254	85	198	28	266	89	207	30	189	63	147	21	248	83	193	28	97	32	76	11	120	40	93	13	98															
17	197	66	153	22	1525	508	1186	169	1129	376	878	125	1140	380	887	127	681	227	530	76	41	14	32	5																
18	98	33	76	11	763	254	593	85	564	188	439	63	570	190	443	63	341	114	265	38	21	8	17																	
19	147	49	115	16	1144	381	890	127	847	282	658	94	855	285	665	95	511	170	398	57	44	24																		
20	49	16	38	5	381	127	297	42	282	94	219	31	285	95	222	32	170	57	133	19	23																			
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22	136	45	106	15	444	148	345	49	473	158	368	53	111	37	86	12	272	97	214																					
23	204	68	159	23	666	222	518	74	710	237	552	79	166	55	129	18	569	306																						
24	68	23	53	8	222	74	173	25	237	79	184	26	55	18	43	6	292																							
25	100	33	78	11	623	208	485	69	1247	416	970	139	246	82	192	27																								
26	50	17	39	6	312	104	242	35	623	208	485	69	128	46	100																									
27	75	25	58	8	468	156	364	52	935	312	727	104	267	144																										
28	25	8	19	3	156	52	121	17	312	104	242	35	137																											
30	122	41	95	14	504	168	392	56	575	206	452																													
31	182	61	142	20	755	252	588	84	1202	647																														
32	61	20	47	7	252	84	196	28	617																															
33	564	188	439	63	407	136	317	45																																
34	282	94	219	31	211	75	166																																	
35	423	141	329	47	441	238																																		
36	141	47	110	16	226																																			
37	371	124	289	41																																				
38	193	69	151																																					
39	402	217																																						
40	206																																							

Appendix B – Detailed Asset Model

The model for **inflation** is

$dq_t = {}^q\Phi_3 dq_{t-1} + (1 - {}^q\theta_1 B - {}^q\theta_2 B_2) * {}^q\varepsilon_t$

and, $q_t = q_{t-1} + dq_t$

and $Q_t = Q_{t-1} * \exp(q_t)$

Where dq_t = change in force of inflation over quarter t, happening immediately at the start of quarter t,

q_t = force of inflation per quarter applying over quarter t, from time t-1 to t

Q_t = CPI index at end of quarter t

And, ${}^q\varepsilon_t$=i.i.d.N(0, ${}^qs^2$)

The model for **short-term yield**

$dn_t = B({}^n\omega_1 - {}^n\omega_2 B)dq_t + (1 - {}^n\theta_3 B^4) * {}^n\varepsilon_t$

$n_t = n_{t-1} + dn_t$

$N_t = (\exp(n_t) - 1) * 400$

Where, dn_t = change in force of treasury yields over quarter t, happening immediately at the start of quarter t, namely time t-1

n_t = force of treasury yields per quarter applying over quarter t

N_t = Treasury yield over quarter t as % per annum

and, ${}^n\omega_t$=i.i.d.N(0,${}^ns^2$)

The model for **long-term yield** is

$dl_t = {}^l\omega_1 dn_t + {}^l\varepsilon_t$

$l_t = l_{t-1} + dl_t$

$L_t = [\exp(2l_t) - 1] * 200$

and, dl_t= change in force of bond yields over quarter t, happening immediately at start of quarter t, namely time t-1

l_t = force of bond yields over quarter t, from time t-1 to t

Lt = ten year bond yield over quarter t as a nominal per annum rate convertible half yearly

and $^{l}\varepsilon_t$=iid N(0, $^{l}s^2$),

The model for **share price yields**

$\rho_t = {}^{\rho}\Phi_0 + {}^{\rho}\varepsilon_t$

$P_t = P_{t-1} * \exp(\rho_t)$

where,

ρ_t = force of share price yields over quarter t, time t-1 to t

P_t = SPI at end of quarter t, time t

and, $^{\rho}\varepsilon_t$ = i.i.d.N(0, $^{\rho}s^2$),

The model for **share dividends and inflation**

$y_t = {}^{y}\Phi_3 * y_{t-4} + {}^{y}\Phi_0 * (1 - {}^{y}\Phi_3) + {}^{y}\omega_1 q_{t-1} + {}^{y}\omega_1 \, {}^{y}\Phi_3 \, q_{t-5} + {}^{y}\varepsilon_t + {}^{y}\theta_2 \; {}^{y}\varepsilon_{t-1}$

$Y_t = [\exp(y_t) - 1] * 400$

Where y_t = force of share dividend yields over quarter t, time t-1 to t

Y_t= share dividend yield as nominal p.a. convertible quarterly

and, $^{y}\varepsilon_t$=i.i.d.N(0, $^{y}s^2$).