HEALTH INSURANCE: ACTUARIAL ASPECTS

Ermanno Pitacco

University of Trieste (Italy)

ermanno.pitacco@econ.units.it

Agenda

1. The need for health-related insurance covers
2. Products in the area of health insurance
3. Between Life and Non-Life insurance: the actuarial structure of sickness insurance
4. Indexation mechanisms
5. Individual experience rating: some models
6. The (aggregate) longevity risk in lifelong covers

1 THE NEED FOR HEALTH-RELATED INSURANCE COVERS

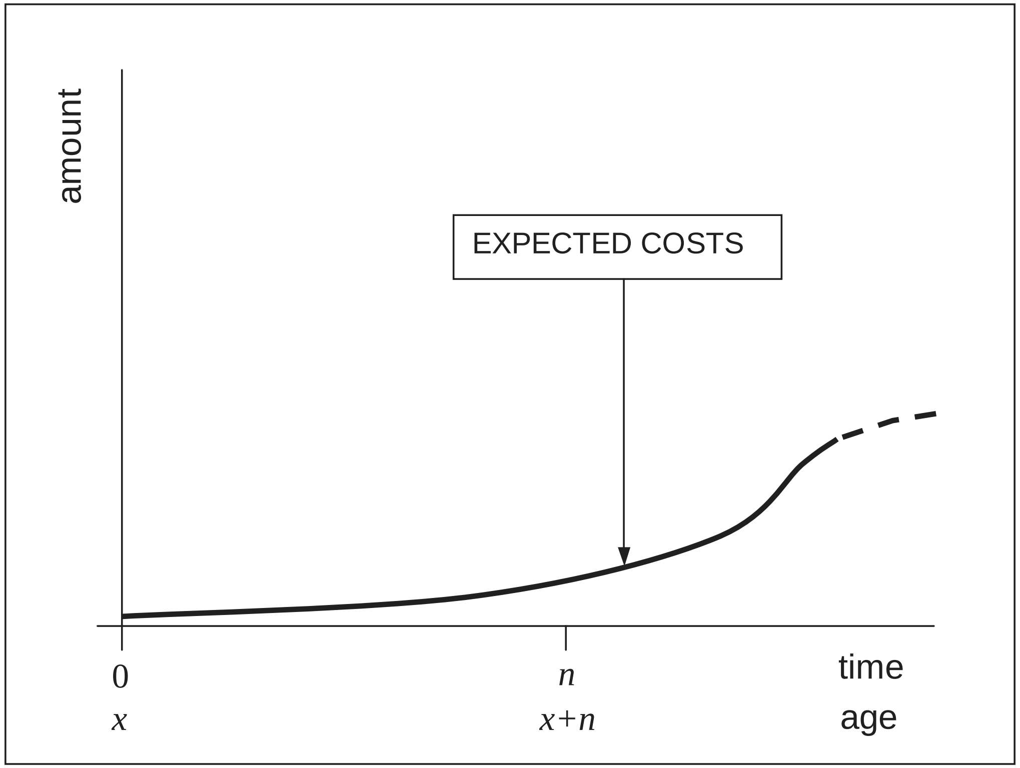
1. Individual flows
2. Aims of health insurance products
3. Risks inherent in the random lifetime

1.1 INDIVIDUAL FLOWS

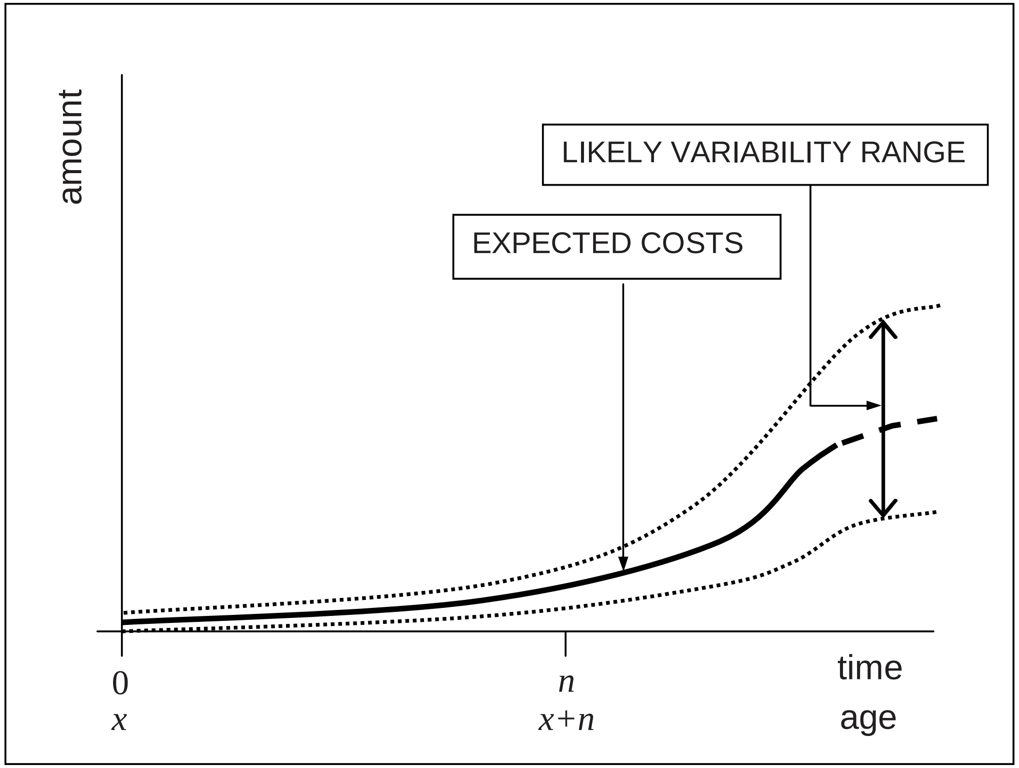
The following flows are considered

- inflows:
  - ▷ earned income (wage / salary)
  - ▷ pension (+ possible life annuities)
- outflows: health-related costs
  - ▷ medical expenses (medicines, hospitalization, surgery, etc.)
  - ▷ expenses related to long-term care
  - ▷ loss of income because of disability (caused by sickness or accident)

Individual flows (cont'd)

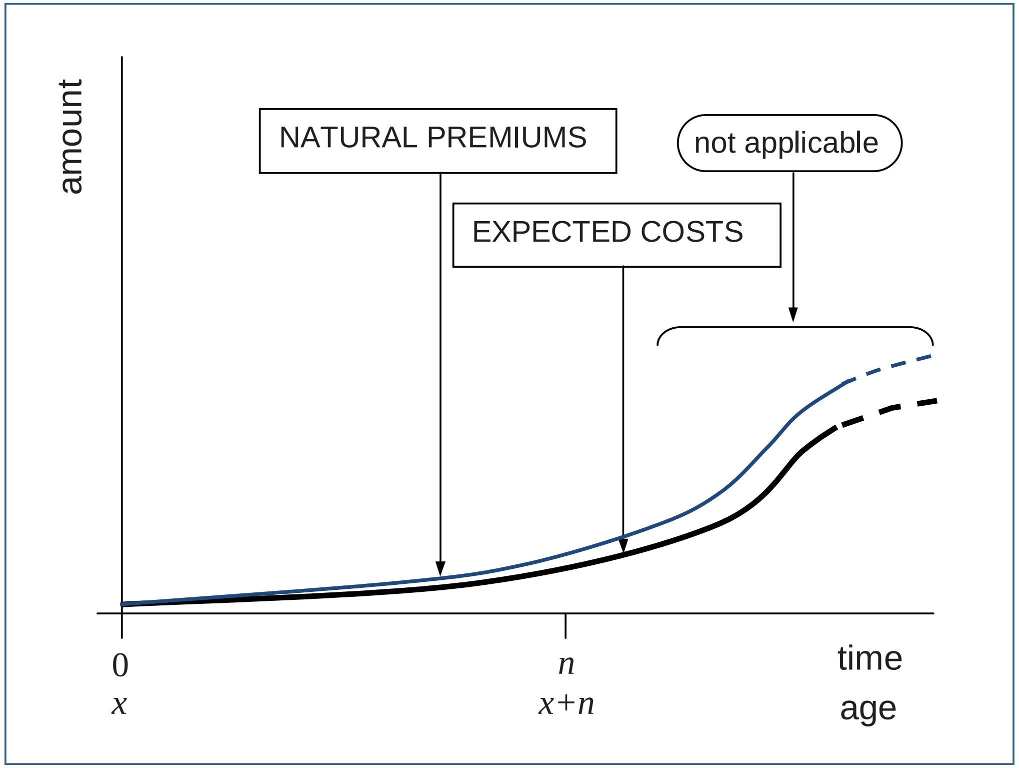


Health-related expected costs

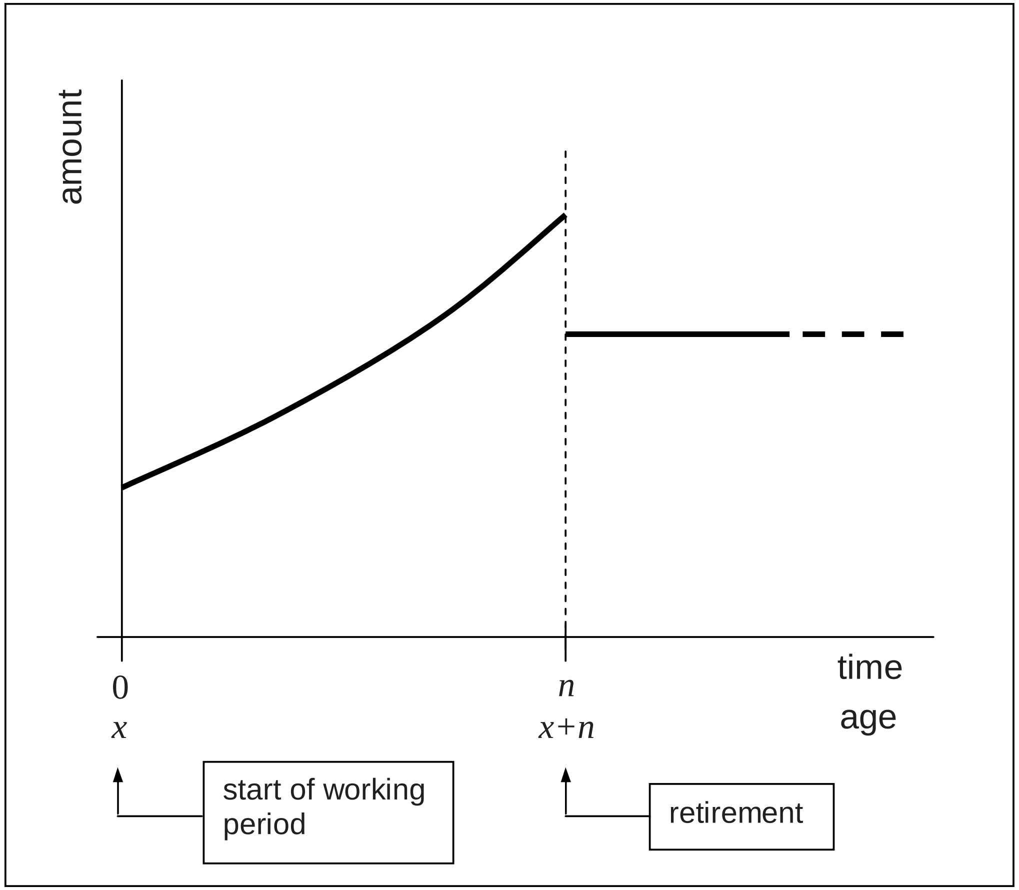


Health-related expected costs and their variability

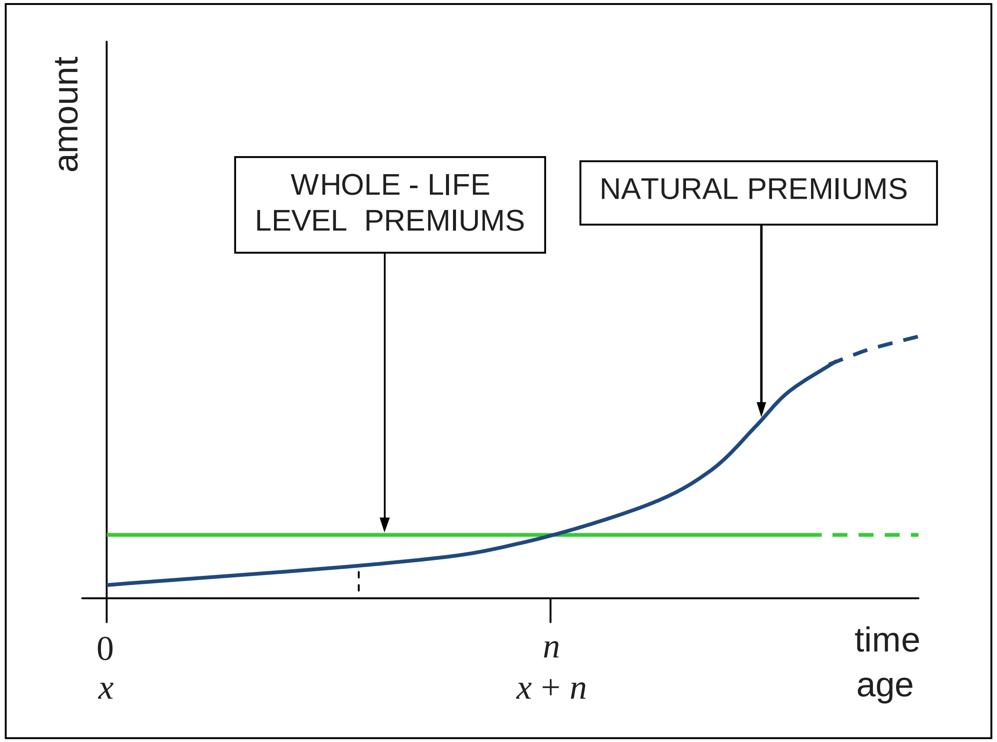
Individual flows (cont'd)



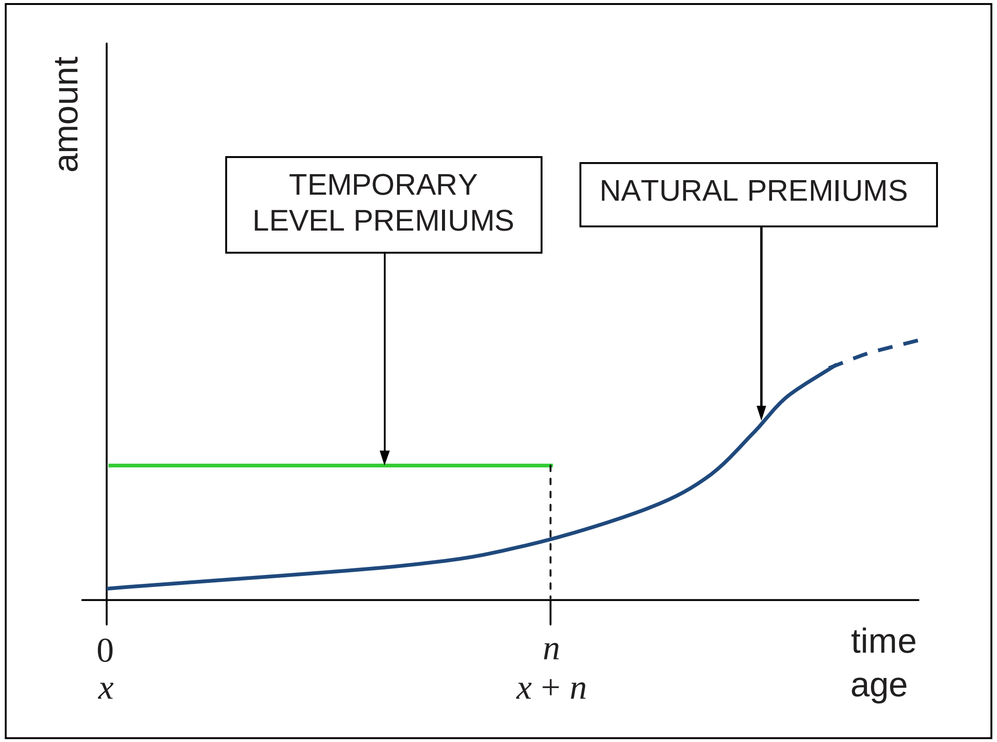
Health-related expected costs and natural premiums (including safety loading)



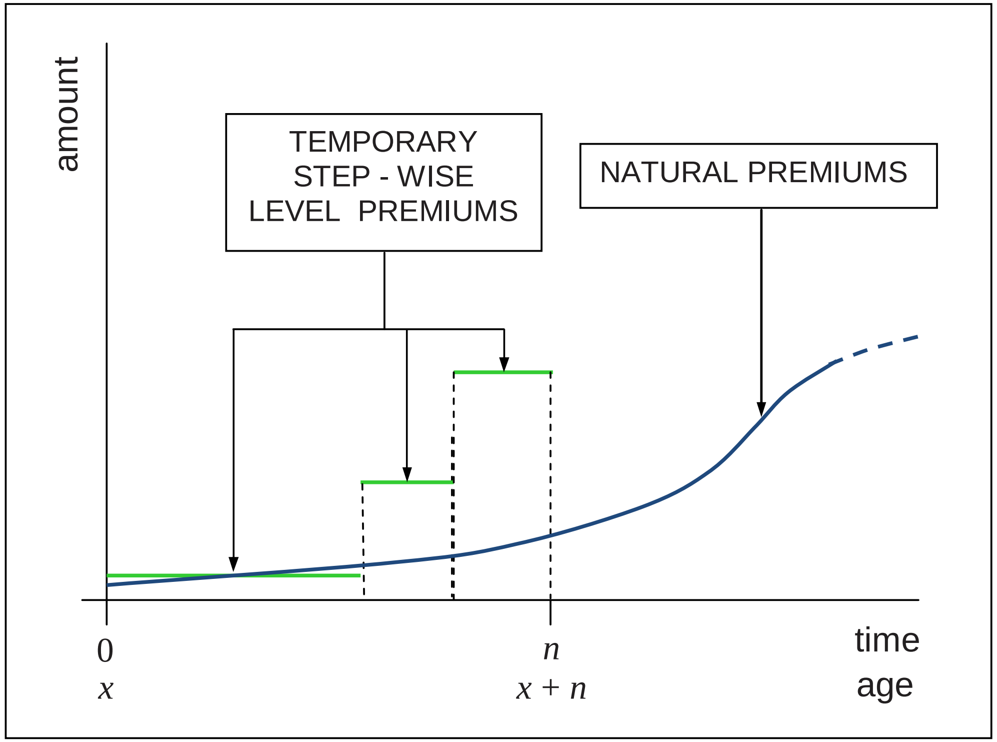
Income profile



Health-related expected costs and whole-life level premiums

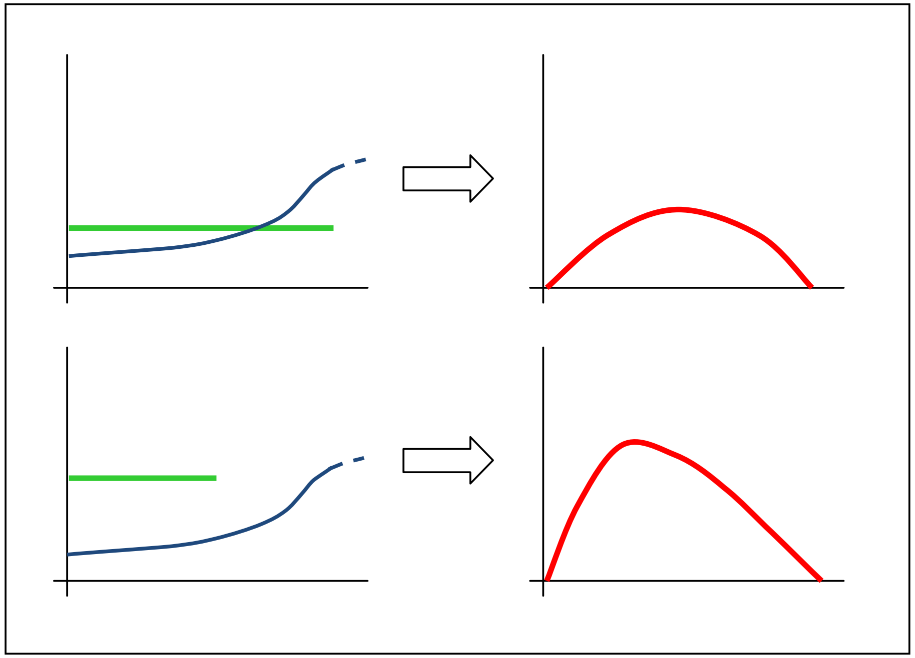


Health-related expected costs and temporary level premiums



Health-related expected costs and temporary step-wise level premiums

Individual flows (cont'd)



Level premiums vs natural premiums, and the reserving process

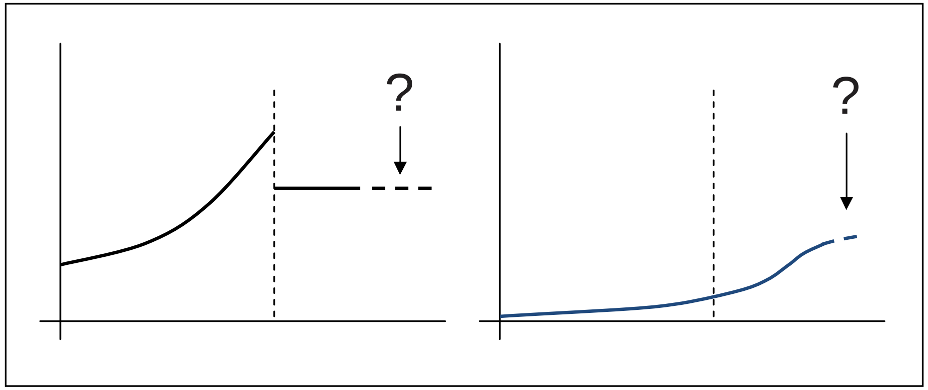
1.2 AIMS OF HEALTH INSURANCE PRODUCTS

1. Replace random costs with deterministic costs (insurance premiums)
   - risk coverage
2. Limit the consequences of time mismatching between income and health costs
   - pre-funding and risk coverage
   - pre-funding $\Rightarrow$ long term products (possibly lifelong)

1.3 RISKS INHERENT IN THE RANDOM LIFETIME

Random lifetime $\Rightarrow$ random duration of

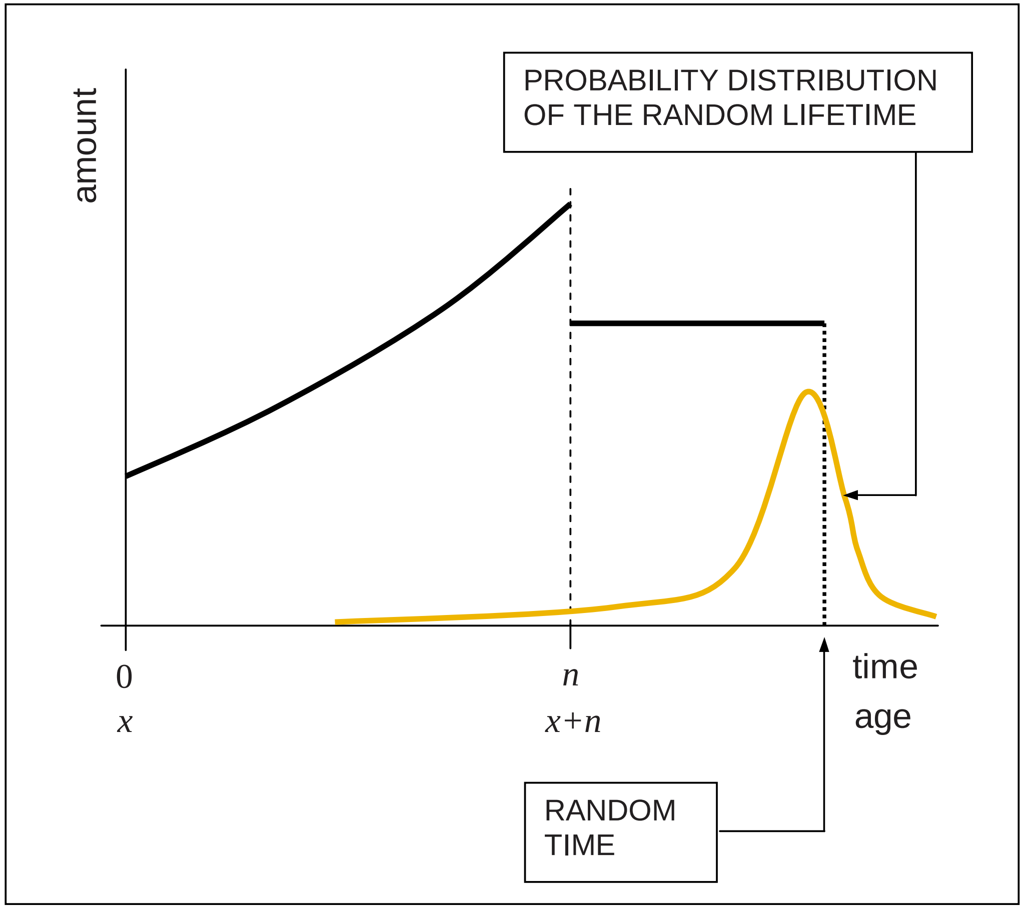
- income (working period and retirement)
- health costs
- premiums



Randomness in lifetime

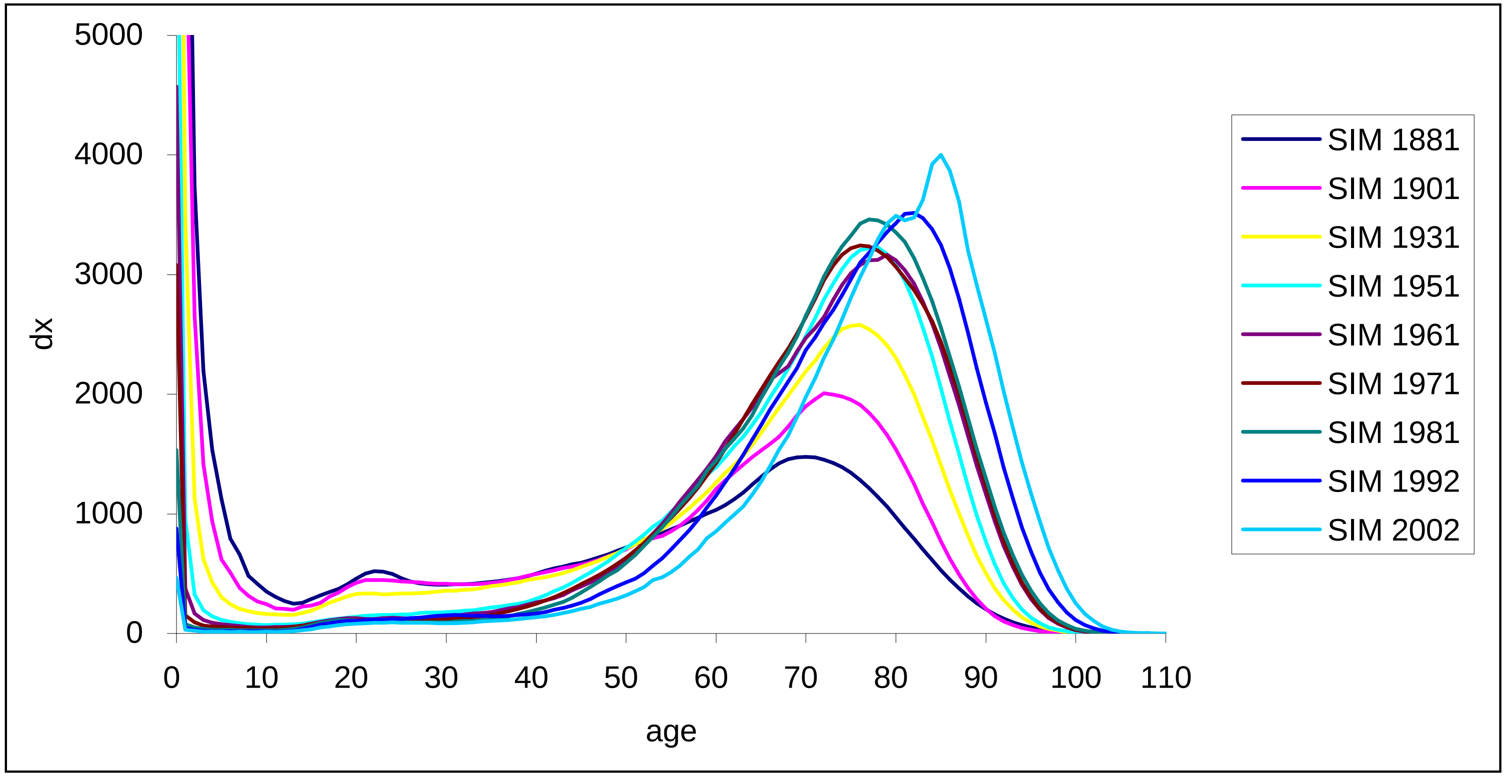
Possible assessment via probability distribution of the lifetime

Risks inherent in the random lifetime (cont'd)



Probability distribution of the random lifetime

Risks inherent in the random lifetime (cont'd)



Probability distributions of the random lifetime (Source: ISTAT - Italian Males)

Risks inherent in the random lifetime (cont'd)

Difficulties originated by coexistence of:

- ▷ random fluctuations of numbers of survivors around expected values
  $\Rightarrow$ *individual longevity risk*

and, more critical:

- ▷ systematic deviations of numbers of survivors from expected values, because of uncertainty in future mortality trend
  $\Rightarrow$ *aggregate longevity risk*

2 PRODUCTS IN THE AREA OF "HEALTH INSURANCE"

1. General aspects
2. Main products

2.1 GENERAL ASPECTS

“Health insurance”: in several countries, a large set of insurance products providing benefits in the case of need arising from:

- *accident*
- *illness*

and leading to:

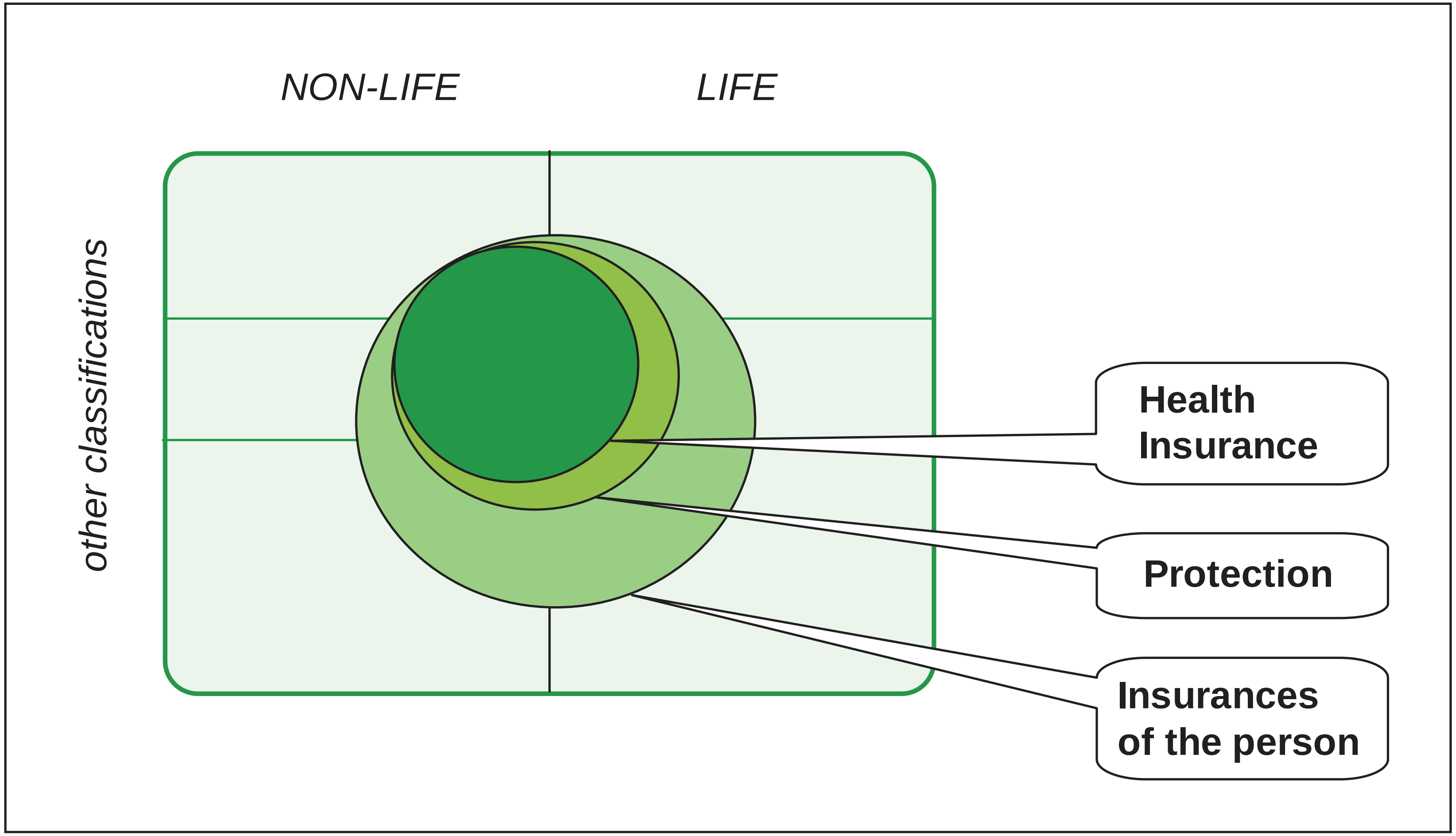
▷ *loss of income* (partial or total, permanent or non-permanent)

▷ *expenses* (hospitalization, medical and surgery expenses, nursery, etc.)

Area: health insurance belongs to the area of *insurances of the person*, which includes

- *life insurance* (in a strict sense): benefits are due depending on death and survival only, i.e. on the insured's lifetime
- *health insurance*: benefits are due depending on the health status, and relevant economic consequences (and depending on the lifetime as well)
- *other* insurances of the person: benefits are due depending on events such as marriage, birth of a child, education and professional training of children, etc.

Health insurance (in broad sense) products are usually shared by “life” and “non-life” branches depending on national legislation and regulation



Health insurance in the context of insurances of the person

2.2 MAIN PRODUCTS

Types of benefits

- *Reimbursement benefit*: to meet (totally or partially) health costs, e.g. medical expenses
- *Forfeiture allowance*: amounts stated at policy issue, e.g. to provide an income when the insured is prevented by sickness or injury from working
  - ▷ annuity
  - ▷ lump sum
- *Service benefit*: care service, e.g. hospital, CCRC (Continuing Care Retirement Communities), etc.

Classification of products

- Accident insurance
- Sickness insurance
- Health benefits as riders to a basic life insurance cover
- Critical Illness (or Dread Disease) insurance
- Disability annuities
- Long Term Care insurance

Remark

In the following (see products listed in Sect. 3.2) we focus on "sickness insurance"

3 BETWEEN LIFE AND NON-LIFE INSURANCE: THE ACTUARIAL STRUCTURE OF SICKNESS INSURANCE

1. Introduction
2. One-year covers
3. Multi-year covers
4. From the basic model to more general models

3.1 INTRODUCTION

Life insurance aspects

mainly concerning medium and long term contracts: disability annuities, LTC insurance, some types of sickness insurance products

- survival modeling

  benefits are due in case of life $\Rightarrow$ to be on the "safe side", survival probabilities should not be underestimated

- financial issues

  asset accumulation (backing technical reserves), return to policyholders

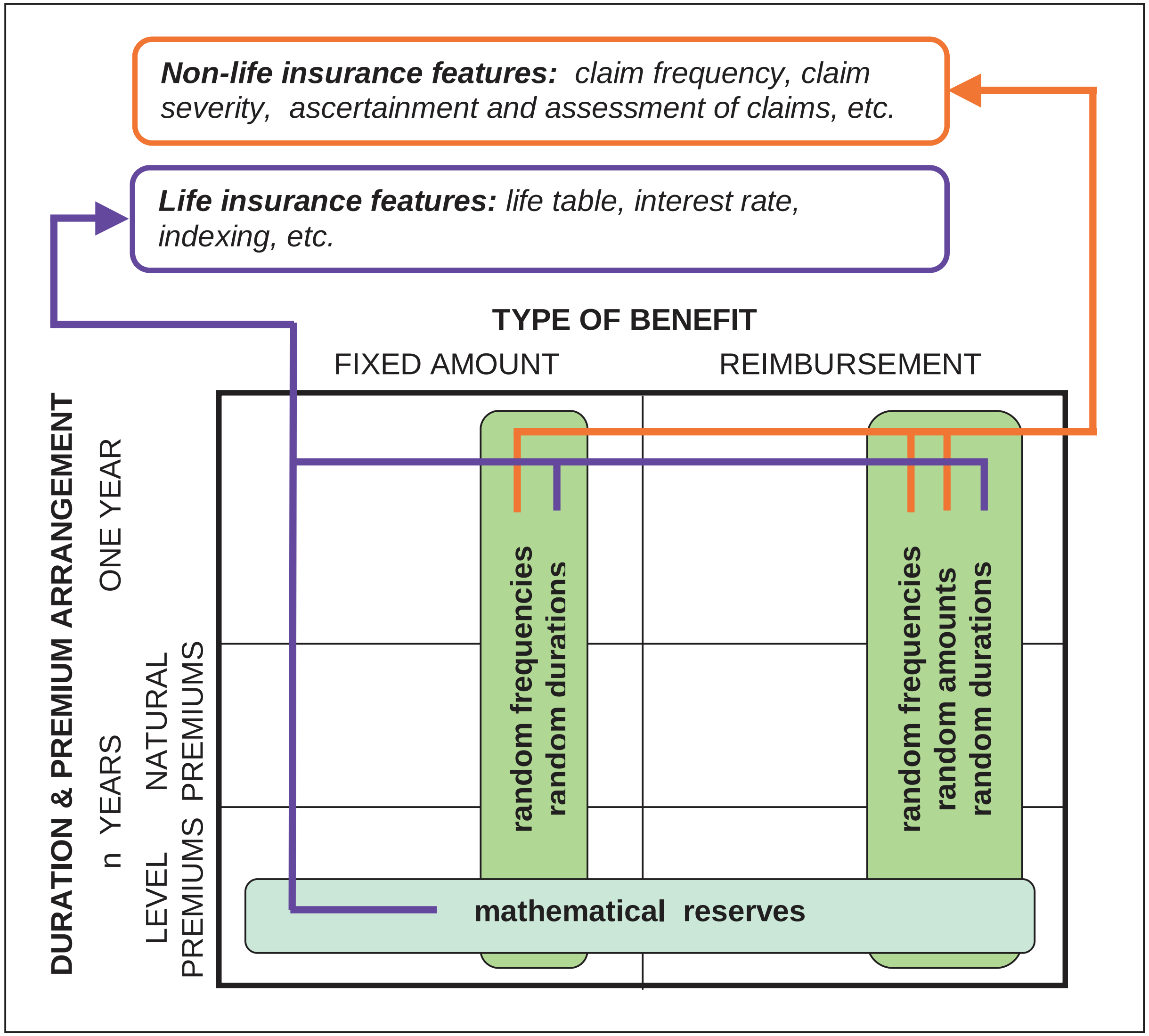
Non-Life insurance aspects

- claim frequency concerns all types of covers

  problems: availability, data format, experience monitoring and experience rating

- claim size concerns insurance covers providing reimbursement (e.g. medical expenses), and covers in which benefits depend on some health-related parameter, e.g. the degree of disability

- expenses
  - ▷ ascertainment and assessment of claims
  - ▷ checking the health status in case of non-necessarily permanent disability



"Life" and "Non-life" aspects in health insurance products

3.2 ONE-YEAR COVERS

Products

1. medical expense reimbursement
2. forfeiture daily allowance for hospitalization
3. forfeiture daily allowance for short-term disability

General features

- Random number N of claims for the generic insured ($N = 0, 1, \dots$)
- Insurer's payment: Y_j for the j-th claim
- Total annual payment to the generic insured: S

$$S = \begin{cases} 0 & \text{if } N = 0 \\ Y_1 + Y_2 + \cdots + Y_N & \text{if } N > 0 \end{cases}$$

- *Premium calculation*: equivalence principle
- Net premium

$$\mathit{\Pi} = \mathbb{E}[S]$$

or (to approx take into account timing of payments)

$$\mathit{\Pi} = \mathbb{E}[S]\,(1+i)^{-\frac{1}{2}}$$

where i = interest rate

- Hypotheses (realistic ?)
  - ▷ for any $N = n$, stochastic independence and identical probability distribution or random variables (r.v.) $Y_1, Y_2, \dots, Y_n$
  - ▷ stochastic independence of r.v. $N, Y_1, Y_2, \dots$
- Hypotheses $\Rightarrow$ factorizing the expectation of S

$$\mathbb{E}[S] = \mathbb{E}[Y]\,\mathbb{E}[N]$$

with Y random variable distributed as the Y_j's

Statistical estimation

- Estimate the quantities $\mathbb{E}[Y], \mathbb{E}[N]$ (technical basis)
- Assumption: "analogous" risks, in terms of amounts (maximum amounts) and exposure time
- Portfolio of *medical expense reimbursement policies*
  - ▷ data
    - ○ r = number of insured risks
    - ○ m = number of claims in the portfolio
    - ○ $y_1, y_2, \dots, y_m$ = amounts paid
  - ▷ average claim amount per claim

$$\bar{y} = \frac{y_1 + y_2 + \cdots + y_m}{m}$$

  - ▷ average number of claims per policy ("claim frequency" index)

$$\phi = \frac{m}{r}$$

- ▷ estimates: $\phi \to \mathbb{E}[N]$, $\bar{y} \to \mathbb{E}[Y]$
- ▷ premium

$$\Pi = \bar{y}\,\phi\,(1+i)^{-\frac{1}{2}}$$

- Portfolio of *forfeiture daily allowance policies*
  - ▷ data
    - ○ r = number of insured risks
    - ○ m = number of claims in the portfolio
    - ○ $g_1, g_2, \ldots, g_m$ = claim lengths in days
  - ▷ average length per claim

$$\bar{g} = \frac{g_1 + g_2 + \cdots + g_m}{m}$$

  - ▷ average number of claims per policy ("claim frequency" index)

$$\phi = \frac{m}{r}$$

- ▷ estimates: $\phi \to \mathbb{E}[N]$, $\bar{g} \to \mathbb{E}[Y]$ (for a unitary daily allowance)
- ▷ premium (for a daily allowance d)

$$\Pi = d\,\bar{g}\,\phi\,(1+i)^{-\frac{1}{2}}$$

- ▷ *morbidity coefficient* = average length of claim per policy

$$\bar{g}\,\phi = \frac{g_1 + g_2 + \cdots + g_m}{r}$$

- A more general (and realistic) setting $\Rightarrow$ allowing for:
  - ▷ amounts exposed to risk (annual maximum amounts)
  - ▷ exposure time (within 1 observation year)

Risk factors

Split a population into risk classes, according to values assumed by risk factors

Risk factors

- objective: physical characteristics of the insured (age, gender, health records, occupation)
- subjective: personal attitude towards health, which determines the individual demand for medical treatments and, consequently, the application for insurance benefits

Incidence of age: see the following Table

Example

x	$100\,\phi_x$	x	$100\,\phi_x$
$15-19$	6.54	$45-49$	11.17
$20-24$	7.13	$50-54$	12.35
$25-29$	5.72	$55-59$	18.71
$30-34$	5.71	$60-64$	19.62
$35-39$	6.23	$65-69$	24.90
$40-44$	10.03		
$100\,\phi = 10.48$			

Average number of claims
as a function of the age; males (Source: ISTAT)
$\phi =$ *overall average*

Premiums

- Age as a risk factor $\Rightarrow$ probability distribution of the random variable S depending on age
- In particular: estimated values $\bar{y}_x$, ϕ_x, $\bar{g}_x$ as functions of age x
- Premiums

$$\Pi_x = \bar{y}_x \, \phi_x \, (1+i)^{-\frac{1}{2}}$$

$$\Pi_x = d \, \bar{g}_x \, \phi_x \, (1+i)^{-\frac{1}{2}}$$

or, considering just the average number of claims as a function of the age

$$\Pi_x = \bar{y} \, \phi_x \, (1+i)^{-\frac{1}{2}}$$

$$\Pi_x = d \, \bar{g} \, \phi_x \, (1+i)^{-\frac{1}{2}}$$

- "Multiplicative" model
  - ▷ Assume

$$\phi_x = \phi\, t_x$$
$$\bar{y}_x = \bar{y}\, u_x$$
$$\bar{g}_x = \bar{g}\, v_x$$

    where
    - quantities $\phi, \bar{y}, \bar{g}$ do not depend on age
    - coefficients t_x, u_x, v_x express the age effect (*aging coefficients*)
  - ▷ Practical interest: assuming that the specific age effect does not change throughout time, claim monitoring can be restricted to quantities $\phi, \bar{y}, \bar{g}$ observed over the whole portfolio $\Rightarrow$ more reliable estimates

Example

Forfeiture daily allowance ($d = 100$)

Assumptions (ISTAT data, graduated by ANIA):

$$\phi_x = \underbrace{0.1048}_{\phi} \times \underbrace{0.272859 \times \mathrm{e}^{0.029841\,x}}_{t_x}$$

$$\bar{g}_x = \underbrace{10.91}_{\bar{g}} \times \underbrace{0.655419 \times \mathrm{e}^{0.008796\,x}}_{v_x}$$

x	ϕ_x	$\bar{g}_x$	Π_x
30	0.07000	9.30991	64.213
35	0.08126	9.72849	77.897
40	0.09434	10.16590	94.497
45	0.10952	10.62298	114.635
50	0.12714	11.10060	139.065
55	0.14760	11.59970	168.700
60	0.17135	12.12124	204.651
65	0.19892	12.66623	248.264
70	0.23093	13.23572	301.171

Average number of claims,
average time (days) per claim,
equivalence premium

3.3 MULTI-YEARS COVERS

Premiums

Medical expense reimbursement or forfeiture daily allowance

Age x at policy issue, term m years

Single premium

$$\Pi_{x,m\rceil} = \sum_{h=0}^{m-1} {}_hp_x \, (1+i)^{-h} \, \Pi_{x+h}$$

with ${}_hp_x$ probability, for a person age x, of being alive at age $x+h$

Natural premiums: $\Pi_x, \Pi_{x+1}, \dots, \Pi_{x+m-1}$, with

$$\Pi_x < \Pi_{x+1} < \cdots < \Pi_{x+m-1}$$

(see table above)

Single premium in a multiplicative model

For example, if

$$\Pi_x = \bar{y}_x \, \phi_x \, (1+i)^{-\frac{1}{2}} = \bar{y} \, \phi \, u_x \, t_x \, (1+i)^{-\frac{1}{2}}$$

then

$$\begin{aligned}
\Pi_{x,m\rceil} &= \sum_{h=0}^{m-1} {}_h p_x \, (1+i)^{-h} \, \bar{y}_{x+h} \, \phi_{x+h} \, (1+i)^{-\frac{1}{2}} \\
&= \underbrace{\bar{y} \, \phi}_{K \text{ (indep. of age)}} \sum_{h=0}^{m-1} \underbrace{{}_h p_x \, (1+i)^{-h-\frac{1}{2}} \, u_{x+h} \, t_{x+h}}_{w_{x,h} \text{ (dependent on age)}} \\
&= K \sum_{h=0}^{m-1} w_{x,h} \\
&= K \, \pi_{x,m\rceil}
\end{aligned}$$

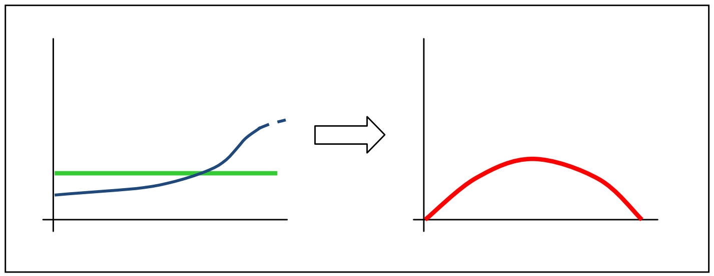
Annual level premium (payable for m years)

$$P_{x,m\rceil} = \frac{\Pi_{x,m\rceil}}{\ddot{a}_{x:m\rceil}}$$

we have

$$P_{x,m\rceil} = \frac{\sum_{h=0}^{m-1} {}_hp_x\,(1+i)^{-h}\,\Pi_{x+h}}{\sum_{h=0}^{m-1} {}_hp_x\,(1+i)^{-h}}$$

thus: annual level premium = arithmetic weighted average of the natural premiums
Consequence: mathematical reserve



Annual level premiums vs natural premiums, and mathematical reserve

Example

Hospitalization daily benefit

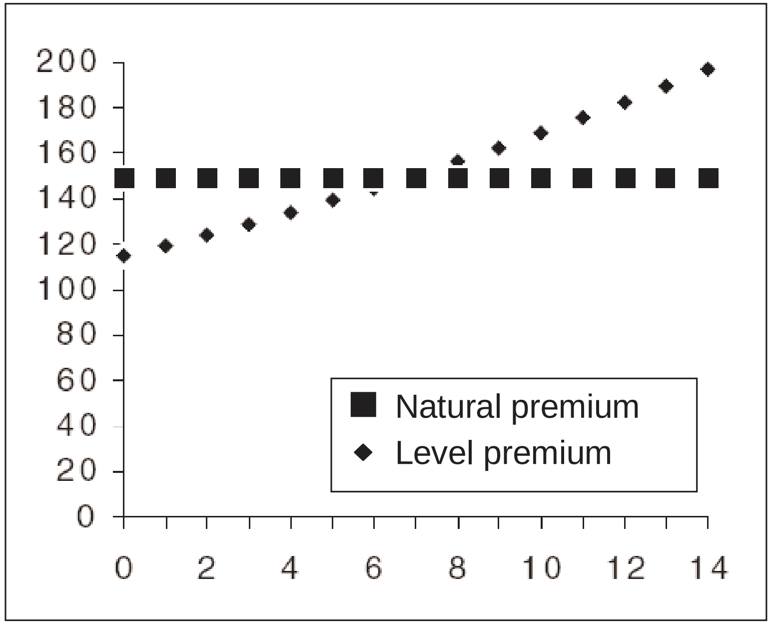
Data: SIM1992; $i = 0.03$; $d = 100$; ϕ_x, $\bar{g}_x$ as above

x	$m = 5$	$m = 10$	$m = 15$	$m = 20$
30	325.944	664.419	1 015.590	1 378.402
35	395.439	805.711	1 229.582	1 663.801
40	479.337	974.563	1 481.880	1 994.168
45	580.127	1 174.416	1 774.530	2 364.920
50	700.958	1 408.786	2 105.144	2 763.054
55	844.022	1 674.369	2 458.869	–
60	1 011.197	1 966.560	–	–
65	1 203.975	–	–	–

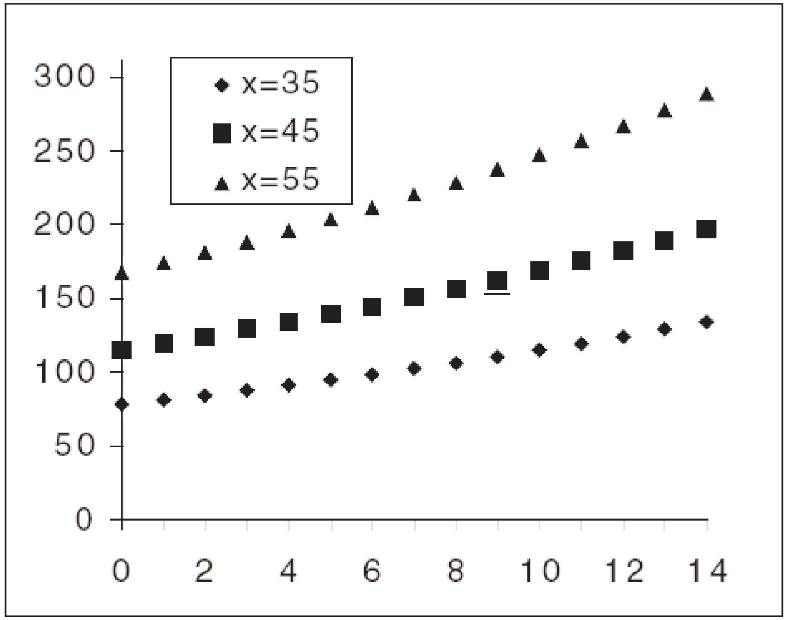
Single premiums

x	$m = 5$	$m = 10$	$m = 15$	$m = 20$
30	69.308	76.120	83.439	91.259
35	84.078	92.337	101.184	110.583
40	101.992	111.984	122.636	133.849
45	123.715	135.776	148.529	161.725
50	150.057	164.560	179.668	194.902
55	181.979	199.300	216.941	–
60	220.649	241.157	–	–
65	267.469	–	–	–

Annual level premiums



Natural premiums and annual level premiums; $x = 45$, $m = 15$



Natural premiums for various ages at policy issue; $m = 15$

Reserves

Prospective mathematical reserve (or *aging reserve*, or *senescence reserve*)

$$V_t = \mathit{\Pi}_{x+t,m-t\rceil} - P_{x,m\rceil}\, \ddot{a}_{x+t:m-t\rceil}; \qquad t = 0, 1, \dots, m \qquad (*)$$

with

$$V_0 = V_m = 0$$

From $(*)$ we find

$$V_t = \mathit{\Pi}_{x+t,1\rceil} - P_{x,m\rceil} + {}_1p_{x+t}\,(1+i)^{-1}\,(\mathit{\Pi}_{x+t+1,m-t-1\rceil} - P_{x,m\rceil}\,\ddot{a}_{x+t+1:m-t-1\rceil})$$

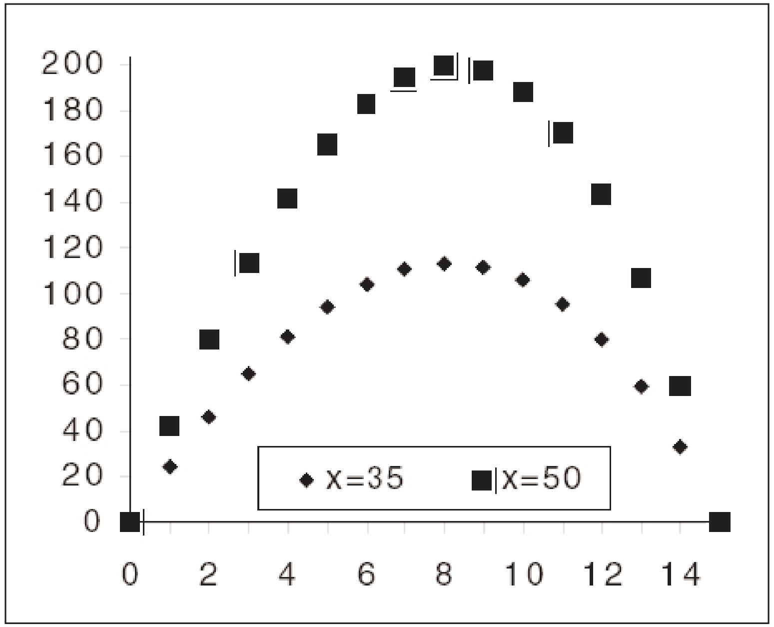
and, as $\mathit{\Pi}_{x+t,1\rceil} = \mathit{\Pi}_{x+t}$, we have the recursion

$$V_t + P_{x,m\rceil} = \mathit{\Pi}_{x+t} + {}_1p_{x+t}\,(1+i)^{-1}\,V_{t+1}$$

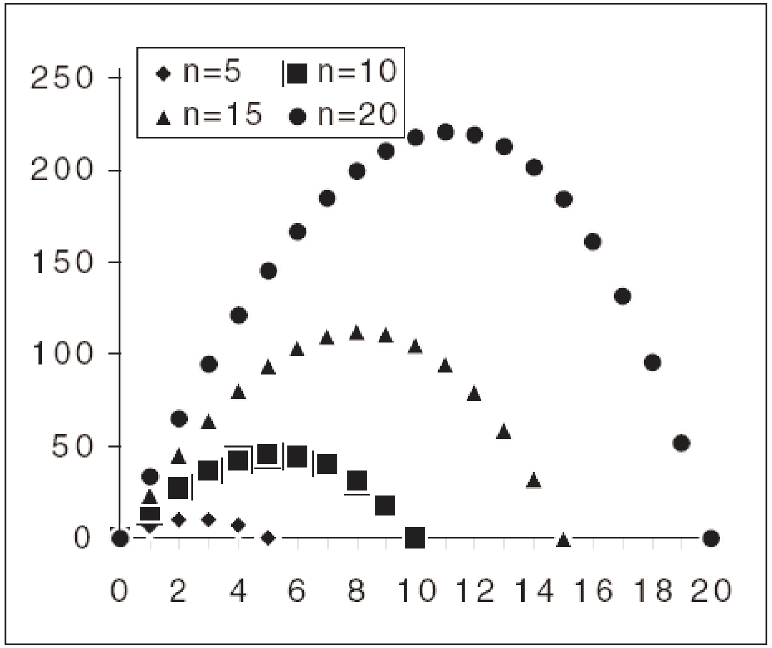
$\Rightarrow$ technical balance in year $(t, t+1)$

Example

Hospitalization daily benefit. Data: as above



Reserves for two ages at policy issue;
$m = 15$



Reserves for various policy terms;
$x = 35$

3.4 FROM THE BASIC MODEL TO MORE GENERAL MODELS

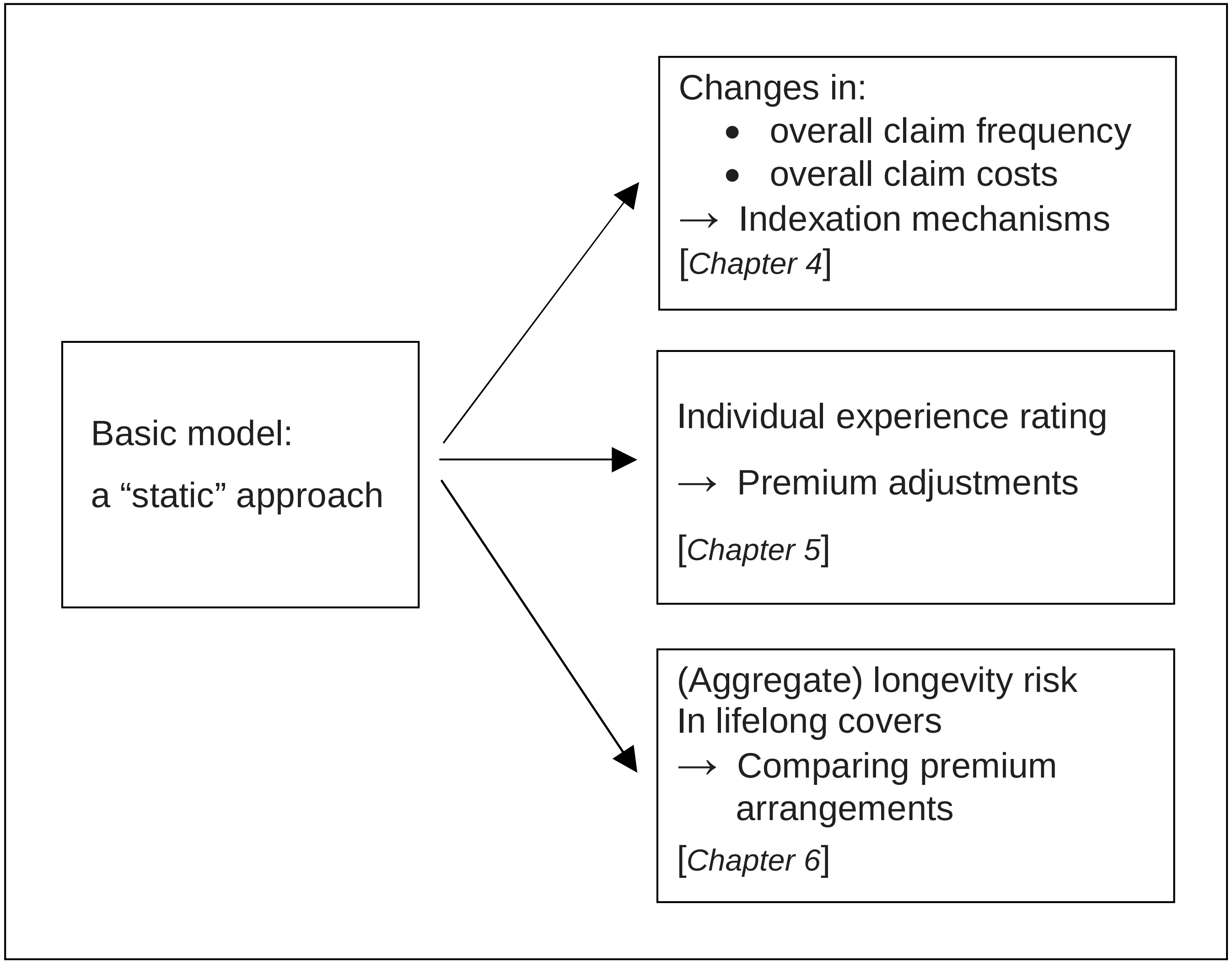
Basic model: a “static” approach, under

- an individual perspective
- a portfolio (or population) perspective

Possible generalizations, in particular allowing for dynamic features:

- ▷ claim frequency and claim cost dynamics at portfolio level
- ▷ individual claim experience
- ▷ longevity dynamics and related consequences in lifelong sickness covers

From the basic model to more general models (cont'd)



Introducing dynamic aspects

4 INDEXATION MECHANISMS

1. Introduction
2. The adjustment model

4.1 INTRODUCTION

Refer, for example, to medical reimbursement policies

Possible changes, at a portfolio level (or population level), in

- claim frequency
- average cost per claim (e.g. because of inflation)

throughout the policy duration

Approaches:

1. change policy conditions, so that the actuarial value of future benefits keeps constant throughout time; in particular
   (a) raise the deductible (if any)
   (b) lower the maximum amount
2. allow for variations in actuarial values of benefits because of change in claim frequency and / or average cost per claim
   $\Rightarrow$ indexing policy elements (future premiums and / or reserve) to keep the equivalence principle fulfilled

In what follows, we focus on approach 2 (assuming increase in the actuarial value of benefits)

Refer, for example, to hospitalization benefits

Interest in keeping constant the purchasing power of the daily allowance; then

- $\Rightarrow$ indexation of benefits
- $\Rightarrow$ need for approach 2

4.2 THE ADJUSTMENT MODEL

Actuarial model

- equivalence at time t (see the definition of the reserve $(^*)$)

$$V_t + P_{x,m\rceil}\, \ddot{a}_{x+t:m-t\rceil} = \mathit{\Pi}_{x+t,m-t\rceil}$$

- assume the multiplicative model

$$\mathit{\Pi}_{x+t,m-t\rceil} = K\, \pi_{x+t,m-t\rceil}$$

- assume that changes only concern the factor K (whilst do not concern the specific effect of age)
- change in the factor

$$K \;\;\Rightarrow\;\; K\,(1 + j^{[\mathrm{K}]})$$

- example: medical expense reimbursement

$$K = \bar{y}\,\phi$$

  - ▷ change in the average cost per claim because of inflation

$$K = \bar{y}\,\phi \;\;\Rightarrow\;\; K\,(1 + j^{[\mathrm{K}]}) = \underbrace{\bar{y}\,(1 + j^{[\mathrm{K}]})}\;\phi$$

- example: hospitalization benefit (daily allowance)

$$K = d\,\bar{g}\,\phi$$

  - ▷ change in the daily allowance to keep the purchasing power

$$K = d\,\bar{g}\,\phi \;\;\Rightarrow\;\; K\,(1 + j^{[\mathrm{K}]}) = \underbrace{d\,(1 + j^{[\mathrm{K}]})}\;\bar{g}\,\phi$$

- change in the actuarial value

$$\mathit{\Pi}_{x+t,m-t\rceil} \;\Rightarrow\; \mathit{\Pi}_{x+t,m-t\rceil}\,(1+j^{[\mathrm{K}]}) = K\,(1+j^{[\mathrm{K}]})\,\pi_{x+t,m-t\rceil}$$

- new equivalence condition at time t:

$$(V_t + P_{x,m\rceil}\,\ddot{a}_{x+t:m-t\rceil})(1+j^{[\mathrm{K}]}) = \mathit{\Pi}_{x+t,m-t\rceil}\,(1+j^{[\mathrm{K}]}) \qquad (^\circ)$$

  or, in more general terms:

$$V_t\,(1+j^{[\mathrm{V}]}) + P_{x,m\rceil}(1+j^{[\mathrm{P}]})\,\ddot{a}_{x+t:m-t\rceil} = \mathit{\Pi}_{x+t,m-t\rceil}\,(1+j^{[\mathrm{K}]}) \quad (^{\circ\circ})$$

  with $j^{[\mathrm{V}]}, j^{[\mathrm{P}]}$ fulfilling equation $(^\circ)$

- equivalence condition on the increments:

$$V_t\,j^{[\mathrm{V}]} + P_{x,m\rceil}\,j^{[\mathrm{P}]}\,\ddot{a}_{x+t:m-t\rceil} = \mathit{\Pi}_{x+t,m-t\rceil}\,j^{[\mathrm{K}]} \qquad (^{\circ\circ\circ})$$

- from $(\circ\circ\circ)$ we find:

$$j^{[K]} = \frac{V_t \, j^{[V]} + P_{x,m\rceil} \, j^{[P]} \, \ddot{a}_{x+t:m-t\rceil}}{\Pi_{x+t,m-t\rceil}}$$

  and then:

$$j^{[K]} = \frac{V_t \, j^{[V]} + P_{x,m\rceil} \, j^{[P]} \, \ddot{a}_{x+t:m-t\rceil}}{V_t + P_{x,m\rceil} \, \ddot{a}_{x+t:m-t\rceil}}$$

  $\Rightarrow$ relation among the three adjustment rates: $j^{[K]}$ is the weighted arithmetic mean of $j^{[V]}, j^{[P]}$

- usually, application of $(\circ\circ\circ)$ each year, to express an annual adjustment of the actuarial value of the insured benefits
  $\Rightarrow$ adjustment rates at time t:

$$j_t^{[K]}, \quad j_t^{[V]}, \quad j_t^{[P]}$$

- in pratice:
  - ▷ increase in the reserve (rate $j_t^{[V]}$) financed by the insurer (profit participation)
  - ▷ increase in premiums (rate $j_t^{[P]}$) paid by the policyholder
- in general:
  - ▷ if $j_t^{[V]} < j_t^{[K]} \quad \Rightarrow \quad j_t^{[P]} > j_t^{[K]}$
  - ▷ if $j_t^{[P]} < j_t^{[K]} \quad \Rightarrow \quad j_t^{[V]} > j_t^{[K]}$

  (because $j_t^{[K]}$ is a weighted arithmetic mean of $j_t^{[V]}, j_t^{[P]}$)

Example

Medical expense reimbursement policy
$x = 50, m = 15$
annual level premiums payable for the whole policy duration

The adjustment model (cont'd)

t	$j_t^{[\mathrm{K}]}$	$j_t^{[\mathrm{V}]}$	$j_t^{[\mathrm{P}]}$
1	0.00086	0.05	0
2	0.00174	0.05	0
3	0.00263	0.05	0
4	0.00355	0.05	0
5	0.00449	0.05	0
6	0.00544	0.05	0
7	0.00641	0.05	0
8	0.00739	0.05	0
9	0.00839	0.05	0
10	0.00939	0.05	0
11	0.01040	0.05	0
12	0.01142	0.05	0
13	0.01244	0.05	0
14	0.01346	0.05	0

Table 1 - Benefit adjustment maintained via reserve increment only

The adjustment model (cont'd)

t	$j_t^{[K]}$	$j_t^{[V]}$	$j_t^{[P]}$
1	0.06	0	0.06105
2	0.06	0	0.06204
3	0.06	0	0.06299
4	0.06	0	0.06389
5	0.06	0	0.06473
6	0.06	0	0.06553
7	0.06	0	0.06626
8	0.06	0	0.06697
9	0.06	0	0.06763
10	0.06	0	0.06823
11	0.06	0	0.06879
12	0.06	0	0.06930
13	0.06	0	0.06977
14	0.06	0	0.07020

Table 2 - Only premium increment to maintain a given benefit adjustment

The adjustment model (cont'd)

t	$j_t^{[K]}$	$j_t^{[V]}$	$j_t^{[P]}$
1	0.06	0.04	0.06036
2	0.06	0.04	0.06069
3	0.06	0.04	0.06104
4	0.06	0.04	0.06138
5	0.06	0.04	0.06171
6	0.06	0.04	0.06204
7	0.06	0.04	0.06236
8	0.06	0.04	0.06268
9	0.06	0.04	0.06300
10	0.06	0.04	0.06330
11	0.06	0.04	0.06360
12	0.06	0.04	0.06389
13	0.06	0.04	0.06418
14	0.06	0.04	0.06445

Table 3 - Premium increment, given the reserve increment, to maintain a chosen benefit adjustment

Remark

Sickness insurance policies (in particular temporary policies) are not "accumulation" products $\Rightarrow$ the mathematical reserve is small (see numerical examples in the previous section), provided that the policy duration is not too long

Then:

- ▷ the only increment of the reserve cannot maintain the raise in the actuarial value of future benefits (see Table 1)
- ▷ the raise in the actuarial value of future benefits can be financed by a reasonable increment of future premiums only (see Table 2)

5 INDIVIDUAL EXPERIENCE RATING: SOME MODELS

see:

E. Pitacco (1992), Risk classification and experience rating in sickness insurance, *Transactions of the 24th International Congress of Actuaries*, Montreal, vol. 3: 209-221

1. Introduction
2. The inference model
3. The experience-rating model
4. Some particular rating systems
5. Numerical examples

5.1 INTRODUCTION

In several countries, many policies provide a one-year cover

The insurer is not obliged to renew the policy

In the case of (too many) claims $\Rightarrow$ no renewal

What is better: no cover or higher (experience-based) premium ?

Ratemaking according to individual characteristics

- ▷ a-priori classification

  based on observable risk factors (age, current health conditions, profession, gender (?), . . .)

- ▷ experience-based classification

  claim experience providing information, in order to partially “replace” risk characteristics which are unobservable at policy issue

In this chapter we define:

- a Bayesian inference model fitting the particular characteristics of sickness insurance (see Sect. 5.2), which in particular provides a "straight" experience rating model (Sect. 5.3)
- some practical rating systems (see Sect. 5.4), such as Bonus Malus (BM) and No-Claim Discount (NCD), relying on the inference model

5.2 THE INFERENCE MODEL

Notation

- x = insured's age at policy issue, i.e. time 0
- m = policy term
- N_{x+h} = random number of claims between age $x+h$ and $x+h+1$, $h = 0, 1, \dots, m-1$
- $N_x(k) = \sum_{h=0}^{k-1} N_{x+h}$ = cumulated random number of claims up to time k
- Θ = random parameter in the probabilistic structure of $N_x, N_{x+1}, \dots, N_{x+m-1}$
- θ = generic outcome of Θ

Hypotheses

- given $\Theta = \theta$, the random numbers $N_x, N_{x+1}, \ldots, N_{x+m-1}$ are independent ($\Rightarrow$ conditional independence)
- the probability distribution of N_{x+h}, $h = 0, 1, \ldots, m-1$, is Poisson with parameter $t_{x+h}\,\theta$, briefly $\mathrm{Pois}(t_{x+h}\,\theta)$:

$$\mathbb{P}[N_{x+h} = n|\Theta = \theta] = \mathrm{e}^{-t_{x+h}\theta}\,\frac{(t_{x+h}\,\theta)^n}{n!}; \quad n = 0, 1, \ldots$$

then:

$$\mathbb{E}[N_{x+h}|\Theta = \theta] = t_{x+h}\,\theta$$

$\Rightarrow$ t_{x+h} expresses the age effect; in practice

$$t_x < t_{x+1} < t_{x+2} < \ldots$$

- the probability distribution of Θ is Gamma with given (positive) parameters α, β, briefly $\mathrm{Gamma}(\alpha, \beta) \;\Rightarrow\;$ probability density function (pdf) given by

$$g(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\,\theta^{\alpha-1}\,\mathrm{e}^{-\beta\,\theta}$$

  with

$$\mathbb{E}[\Theta] = \frac{\alpha}{\beta}$$

$$\mathbb{V}\mathrm{ar}[\Theta] = \frac{\alpha}{\beta^2}$$

Some results

- Unconditional distribution of $N_{x+h}, \; h = 0, 1, \ldots, m-1$

$$\begin{aligned}\mathbb{P}[N_{x+h} = n] &= \int_0^{+\infty} \mathbb{P}[N_{x+h} = n | \Theta = \theta] \, g(\theta) \, \mathrm{d}\theta \\ &= \frac{\left(\frac{\beta}{t_{x+h}}\right)^{\alpha} \Gamma(\alpha + n)}{\Gamma(\alpha) \, n! \left(\frac{\beta}{t_{x+h}} + 1\right)^{\alpha+n}}\end{aligned}$$

that is, a negative binomial:

$$\mathrm{NegBin}\left(\alpha, \frac{\frac{\beta}{t_{x+h}}}{\frac{\beta}{t_{x+h}} + 1}\right)$$

- Then:

$$\mathbb{E}[N_{x+h}] = \frac{\alpha}{\frac{\beta}{t_{x+h}}} = t_{x+h}\,\mathbb{E}[\Theta]$$

$$\mathbb{V}\mathrm{ar}[N_{x+h}] = \frac{\alpha\left(\frac{\beta}{t_{x+h}} + 1\right)}{\left(\frac{\beta}{t_{x+h}}\right)^2}$$

- Given $\Theta = \theta$, the probability distribution of $N_x(k)$ is

$$\mathrm{Pois}\left(\theta \sum_{h=1}^{k} t_{x+h-1}\right)$$

Remark

The expression $\mathbb{E}[N_{x+h}] = t_{x+h}\,\mathbb{E}[\Theta]$ for the expected value corresponds to $\phi_{x+h} = t_{x+h}\,\phi$ used in Chap. 3

- Then, the unconditional distribution of $N_x(k)$ is

$$\mathbb{P}[N_x(k) = n] = \int_0^{+\infty} \mathbb{P}[N_x(k) = n | \Theta = \theta] \, g(\theta) \, \mathrm{d}\theta$$

$$= \frac{\left(\frac{\beta}{\sum_{h=1}^{k} t_{x+h-1}}\right)^{\alpha} \Gamma(\alpha + n)}{\Gamma(\alpha) \, n! \left(\frac{\beta}{\sum_{h=1}^{k} t_{x+h}} + 1\right)^{\alpha+n}}; \quad n = 0, 1, \ldots$$

that is,

$$\mathrm{NegBin}\left(\alpha, \frac{\frac{\beta}{\sum_{h=1}^{k} t_{x+h-1}}}{\frac{\beta}{\sum_{h=1}^{k} t_{x+h-1}} + 1}\right)$$

The inference procedure

- Claim record ($k < m$)

$$n_x,\ n_{x+1},\ \dots, n_{x+k-1}$$

- Posterior distribution of the parameter Θ:

$$\begin{aligned} & g(\theta | n_x, n_{x+1}, \dots, n_{x+k-1}) \\ & \propto g(\theta)\, \mathbb{P}[(N_x = n_x) \wedge (N_{x+1} = n_{x+1}) \wedge \dots \wedge (N_{x+k-1} = n_{x+k-1}) | \Theta = \theta] \\ & \propto \mathrm{e}^{-\theta \left(\beta + \sum_{h=0}^{k-1} t_{x+h}\right)}\, \theta^{\alpha + \sum_{h=0}^{k-1} n_{x+h}\, -1} \end{aligned}$$

that is, $\text{Gamma}\left(\alpha + \sum_{h=0}^{k-1} n_{x+h},\ \beta + \sum_{h=0}^{k-1} t_{x+h}\right)$, with

$$\mathbb{E}[\Theta | n_x,\ n_{x+1},\ \dots, n_{x+k-1}] = \frac{\alpha + \sum_{h=0}^{k-1} n_{x+h}}{\beta + \sum_{h=0}^{k-1} t_{x+h}}$$

- Unconditional distribution of N_{x+j}, $j \geq k$, calculated by using $g(\theta | n_x, n_{x+1}, \ldots, n_{x+k-1})$ (instead of $g(\theta)$)
- In particular:

$$\mathbb{E}[N_{x+j} | n_x, n_{x+1}, \ldots, n_{x+k-1}] = t_{x+j} \, \frac{\alpha + \sum_{h=0}^{k-1} n_{x+h}}{\beta + \sum_{h=0}^{k-1} t_{x+h}} \qquad (^\circ)$$

Remark

- ▷ sufficient statistics given by $\left(\sum_{h=0}^{k-1} t_{x+h}, \, \sum_{h=0}^{k-1} n_{x+h} \right)$
- ▷ Eq. $(^\circ)$ $\Rightarrow$ *credibility formula*

$$\mathbb{E}[N_{x+j} | n_x, n_{x+1}, \ldots, n_{x+k-1}] =$$

$$t_{x+j} \left(\frac{\alpha}{\beta} \, \frac{\beta}{\beta + \sum_{h=0}^{k-1} t_{x+h}} + \frac{\sum_{h=0}^{k-1} n_{x+h}}{\sum_{h=0}^{k-1} t_{x+h}} \, \underbrace{\frac{\sum_{h=0}^{k-1} t_{x+h}}{\beta + \sum_{h=0}^{k-1} t_{x+h}}}_{\text{credibility factor } z_{x,k}} \right)$$

Example 1

Assume:

$$\mathbb{E}[N_y] = 0.034761 \times 1.032044^y \quad (y \geq 20) \qquad (*)$$

Let $x = 40$ = age at policy issue

We find:

h	$\mathbb{E}[N_{40+h}]$
0	0.123
1	0.127
2	0.131
3	0.135
4	0.139
5	$\boxed{0.144}$

Expected number of claims

Example 2

Purpose: to determine the t 's (useful in inference procedures)

We know that

$$\mathbb{E}[N_y] = t_y \, \mathbb{E}[\Theta]$$

Assume y' as reference age, and set $t_{y'} = 1$

Then:

$$t_y = \frac{\mathbb{E}[N_y]}{\mathbb{E}[N_{y'}]}$$

For example, with $y' = 20$ and the assumption $(*)$ we find the following Table

The inference model (cont'd)

y	t_y
20	1.000
25	1.171
30	1.371
35	1.605
40	1.879
45	2.200
50	2.576
55	3.016
60	3.531
65	4.134
70	4.841

Ageing parameters

Example 3

Assume

- parameters of the gamma distribution:

$$\alpha = 1.1; \quad \beta = 16.83977$$

- age at policy issue $x = 40$

We find the following credibility factors:

k	$z_{x,k}$
1	0.100
2	0.185
3	0.257
4	0.319
5	0.373

Credibility factors

We find the following expected values of N_{45}, depending on the previous claim experience

$\sum_{h=0}^{4} n_{40+h}$	$\mathbb{E}[N_{45} \mid n_{40}, \ldots, n_{44}]$
0	0.090
1	0.172
2	0.254
3	0.336
4	0.418
5	0.500
6	0.582
$\ldots$	$\ldots$

Expected number of claims according to claim experience

5.3 THE EXPERIENCE-RATING MODEL

Annual level premium, payable for m years, if no experience rating is adopted

$$P = \frac{\sum_{h=0}^{m-1} {}_h p_x \, (1+i)^{-h} \, \Pi_{x+h}}{\ddot{a}_{x:\overline{m}|}}$$

where, for a medical expenses insurance cover:

$$\Pi_{x+h} = \bar{y} \, \mathbb{E}[N_{x+h}] \, (1+i)^{-\frac{1}{2}}$$

Assuming $\bar{y} = 1$, we have:

$$P = \frac{\sum_{h=0}^{m-1} {}_h p_x \, (1+i)^{-h-\frac{1}{2}} \, \mathbb{E}[N_{x+h}]}{\ddot{a}_{x:\overline{m}|}}$$

(in line with an experience rating system based on the observed number of claims)

In presence of experience rating

- in principle: in every year different premiums should be determined and charged according to each individual claim record
- in practice: a too complex premium system would be generated

To obtain an applicable premium system, we have to state:

- ▷ times at which premium adjustments may occur
- ▷ the number of different premiums at each adjustment time
- ▷ relationships between claim experience and adjusted premiums

See following notation and Figure 1

Notation

- r = number of premium adjustments
- $k_1, \dots, k_r$ = times of premium adjustments; $k = k_1$ if $r = 1$
- $\nu_{\max}$ = number of premiums in the experience rating system
- ν = index of premium ($\nu = 1, 2, \dots, \nu_{\max}$)
- $k(\nu)$ = adjustment time at which premium ν may be charged
- $\sigma(\nu)$ = a set of outcomes of $N_x(k(\nu))$:
  $N_x(k(\nu)) \in \sigma(\nu) \;\Leftrightarrow\;$ premium ν will be charged (at time $k(\nu)$)
- $q(x, h, n) = \mathbb{P}[N_x(h) = n]$ = probability of n claims up to time h
- $s(\nu) = \sum_{n \in \sigma(\nu)} q(x, k(\nu), n)$ = probability that premium ν will be charged (at time $k(\nu)$)
- $P(\nu)$ = amount of premium ν

The experience-rating model (cont'd)

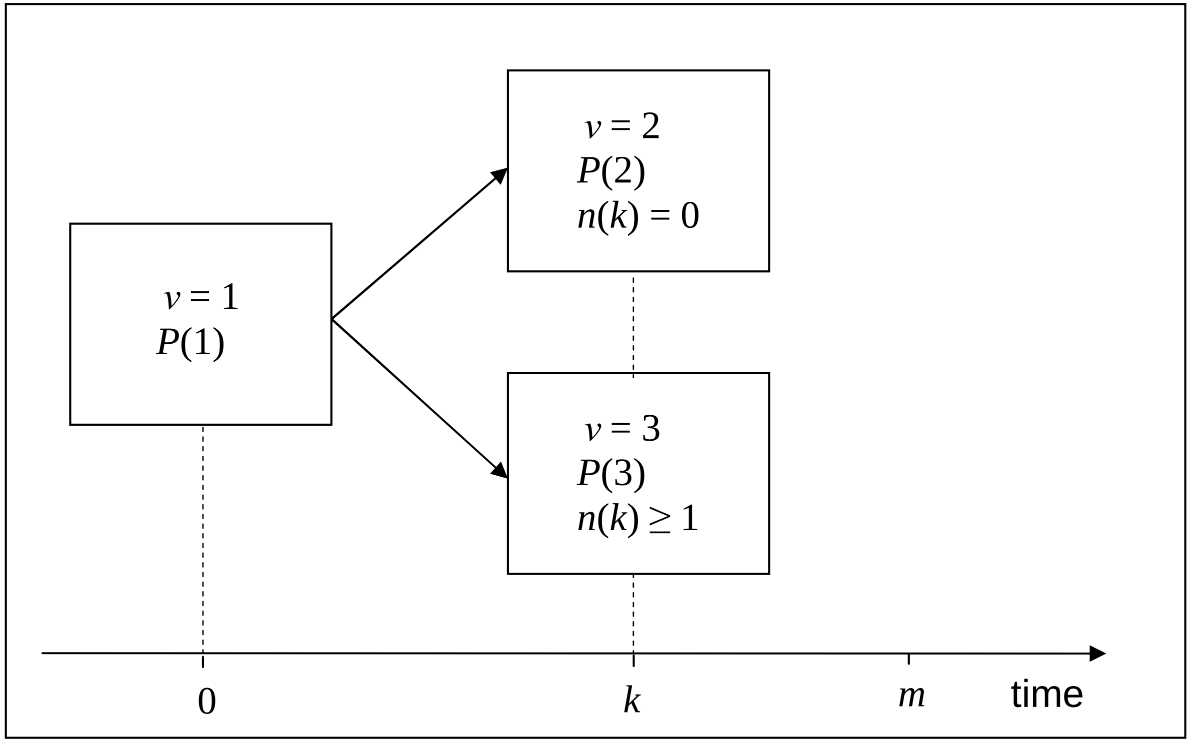


Figure 1 – An experience-based rating system; 1 adjustment time

The experience-rating model (cont'd)

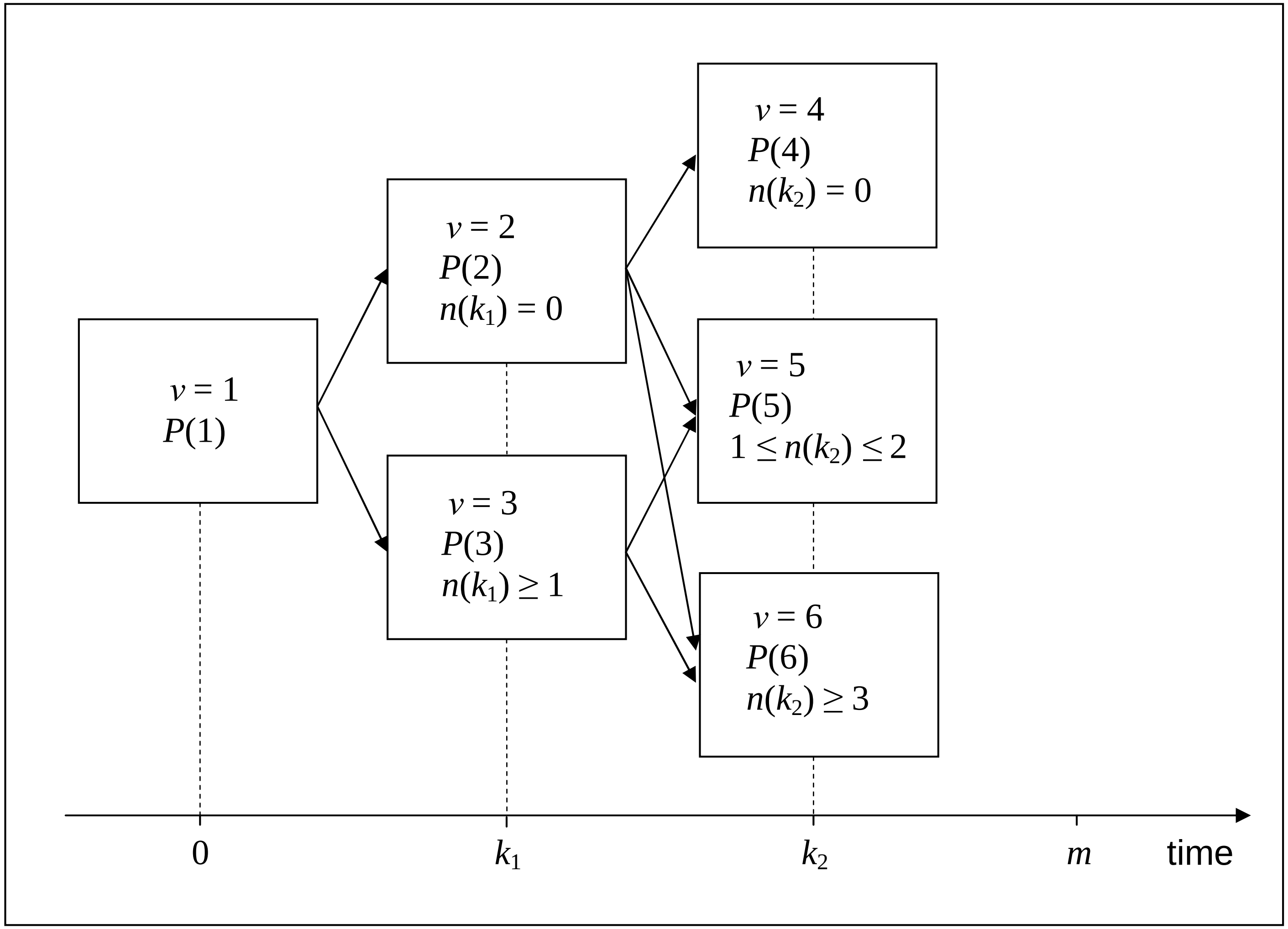


Figure 2 – An experience-based rating system; 2 adjustment times

Premiums

$$P(1) = \frac{\displaystyle\sum_{h=0}^{k_1-1} {}_hp_x \, (1+i)^{-h-\frac{1}{2}} \; \mathbb{E}[N_{x+h}]}{\ddot{a}_{x:k_1\rceil}} \qquad (*)$$

$$P(\nu) = \frac{\displaystyle\sum_{h=k_j}^{k_{j+1}-1} {}_{h-k_j}p_{x+k_j} \, (1+i)^{-h-k_j-\frac{1}{2}} \; \mathbb{E}\Big[N_{x+h} \,|\, \bigvee_{n\in\sigma(\nu)} (N_x(k_j)=n)\Big]}{\ddot{a}_{x+k_j:k_{j+1}-k_j\rceil}} \qquad (**)$$

$\nu = 2, \ldots, \nu_{\max}$; $j = 1, \ldots, r$, with $k_{r+1} = m$

Note that:

- Expected values in $(*)$ calculated before any specific experience; then

$$\mathbb{E}[N_{x+h}] = t_{x+h}\,\mathbb{E}[\Theta] = t_{x+h}\,\frac{\alpha}{\beta}$$

- Conditional expected values in $(**)$ depend on the specific information provided by the adoption of premium $P(\nu)$, i.e. by the set of outcomes of $N_x(k_j)$ which imply $P(\nu)$. We have:

$$\mathbb{E}\Big[N_{x+h}\,|\bigvee_{n\in\sigma(\nu)} N_x(k_j)=n\Big]$$

$$= \sum_{n\in\sigma(\nu)} \mathbb{E}[N_{x+h}|N_x(k_j)=n]\,\frac{q(x,k_j,n)}{\sum_{n\in\sigma(\nu)} q(x,k_j,n)}$$

$$= \frac{1}{s(\nu)} \sum_{n\in\sigma(\nu)} \mathbb{E}[N_{x+h}|N_x(k_j)=n]\,q(x,k_j,n)$$

- As $N_x(k_j) = \sum_{i=0}^{k_j-1} N_{x+i}$, we have (according to $(\circ)$):

$$\mathbb{E}[N_{x+h}|N_x(k_j) = n] = t_{x+h}\,\frac{\alpha + n}{\beta + \sum_{i=0}^{k_j-1} t_{x+i}}$$

By using the equations above, we can calculate

$$P(1),\ P(2),\ \ldots,\ P(\nu_{\max})$$

$\Rightarrow$ experience rating system fully defined

5.4 SOME PARTICULAR RATING SYSTEMS

Let $\Pi_{x,m\rceil}$ denote the single premium for a m-year insurance cover:

$$\Pi_{x,m\rceil} = \sum_{h=0}^{m-1} {}_hp_x \,(1+i)^{-h}\, \Pi_{x+h} = \sum_{h=0}^{m-1} {}_hp_x \,(1+i)^{-h-\frac{1}{2}}\, \mathbb{E}[N_h]$$

It can be proved that the set of premiums $P(1), P(2), \ldots, P(\nu_{\max})$ (see $(^*)$, $(^{**})$ in Sect. 5.3) fulfills the equivalence principle, that is

$$\sum_{\nu=1}^{\nu_{\max}} s(\nu)\, P(\nu)\, \ddot{a}_{x+k_j:k_{j+1}-k_j\rceil} = \Pi_{x,m\rceil}$$

Now consider the $\nu_{\max}$ amounts

$$\bar{P}(1),\ \bar{P}(2),\ \ldots,\ \bar{P}(\nu_{\max})$$

We say that the $\bar{P}(\nu)$ are *equivalence premiums* if and only if they fulfill the equivalence principle, i.e.

$$\sum_{\nu=1}^{\nu_{\max}} s(\nu)\, \bar{P}(\nu)\, \ddot{a}_{x+k_j:k_{j+1}-k_j\rceil} = \Pi_{x,m\rceil} \qquad (^{\circ\circ})$$

Note that:

- A particular solution of $(^{\circ\circ})$ is given by $P(1), P(2), \dots, P(\nu_{\max})$
- Other particular solutions of $(^{\circ\circ})$ can be found by stating specific relationships among the premiums, e.g. in order to smooth the sequences of premiums implied by the various claim records
- For example
  - ▷ set
    $$\bar{P}(\nu) = f_\nu\, \bar{P}(1); \quad \nu = 2, 3, \dots, \nu_{\max}$$
  - ▷ solve $(^{\circ\circ})$ with respect to $\bar{P}(1)$
  - ▷ for given f_ν's, calculate $\bar{P}(2), \dots, \bar{P}(\nu_{\max})$

- Alternative approach
  - ▷ define $\bar{P}$ as a *reference premium* (not necessarily charged to the contract, whatever the node)
  - ▷ set
$$\bar{P}(\nu) = f_\nu \, \bar{P}; \quad \nu = 1, 2, \ldots, \nu_{\max}$$
  - ▷ solve $(^{\circ\circ})$ with respect to $\bar{P}$
  - ▷ for given f_ν's, calculate $\bar{P}(1), \bar{P}(2), \ldots, \bar{P}(\nu_{\max})$
- Any premium system
$$\bar{P}(1), \bar{P}(2), \ldots, \bar{P}(\nu_{\max})$$
(other than $P(1), P(2), \ldots, P(\nu_{\max})$) implies a *solidarity effect* among insureds

Remarks

1. Note that, when the approach based on the reference premium is adopted, we may find, because of the choice of the reference premium $\bar{P}$ and the parameters f's,

$$\bar{P}(1) < P(1)$$

   where $P(1)$ is the initial premium in a straight experience-rating model
   Then
   - ▷ the insured is not fully financed throughout the first period, i.e. $(0, k_1)$
   - ▷ loss in case of lapses

2. As regards the mathematical reserve:
   (a) in the straight experience rating model, the $P(\nu)$'s fulfill the equivalence principle in each period, i.e. $(0, k_1)$, (k_1, k_2), ..., then
      - ▷ a small reserve required in each period because of the annual increase in natural premiums
      - ▷ reserve $= 0$ at times $k_1, k_2, \dots$
   (b) in other experience rating systems, the $\bar{P}(\nu)$'s only ensure the equivalence over the cover period $(0, m)$ considered as a whole, then
      - ▷ a higher reserve may be required in each period
      - ▷ reserve $\neq 0$ at times $k_1, k_2, \dots$

NCD systems

A *no-claim discount (NCD)* system can be defined as a solution of $(^{\circ\circ})$

For example (see Figure 3):

- $r = 1$
- k = time of premium adjustment
- $\nu_{\max} = 3$
- $\bar{P}(1)$ = initial premium
- $\bar{P}(2) = f_2\,\bar{P}(1)$; $\bar{P}(3) = \bar{P}(1)$
- $0 < f_2 < 1$
- $\sigma(2) = \{0\}$; $\sigma(3) = \{1, 2, \ldots\}$

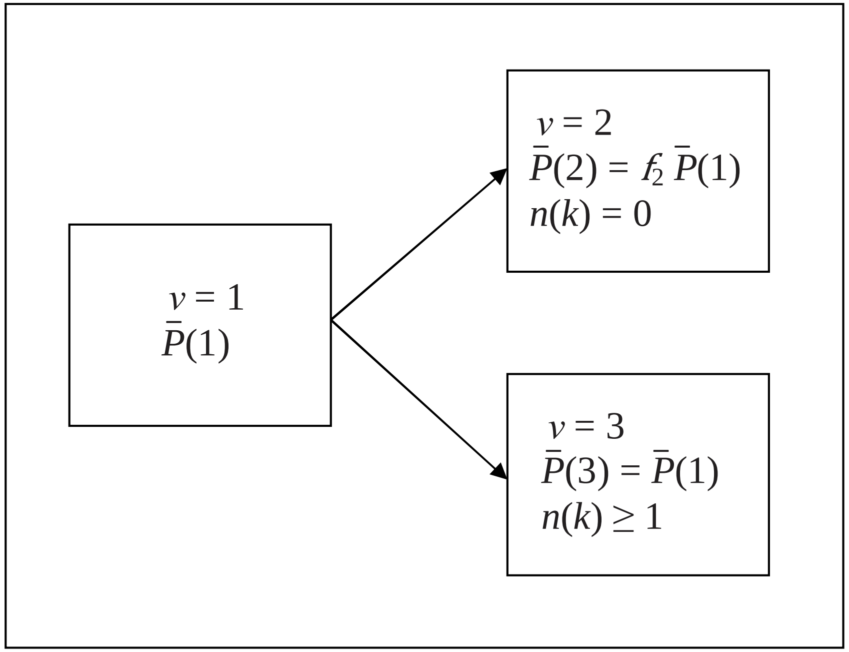


Figure 3 – NCD system: example with 1 adjustment time

Another example (see Figure 4):

- $r = 2$
- k_1, k_2 = times of premium adjustment
- $\nu_{\max} = 5$
- $\bar{P}(1)$ = initial premium
- $\bar{P}(2) = f_2\, \bar{P}(1)$; $\bar{P}(3) = \bar{P}(1)$; $\bar{P}(4) = f_4\, \bar{P}(1)$; $\bar{P}(5) = \bar{P}(1)$
- $0 < f_4 < f_2 < 1$
- $\sigma(2) = \{0\}$; $\sigma(3) = \{1, 2, \dots\}$; $\sigma(4) = \{0\}$; $\sigma(5) = \{1, 2, \dots\}$

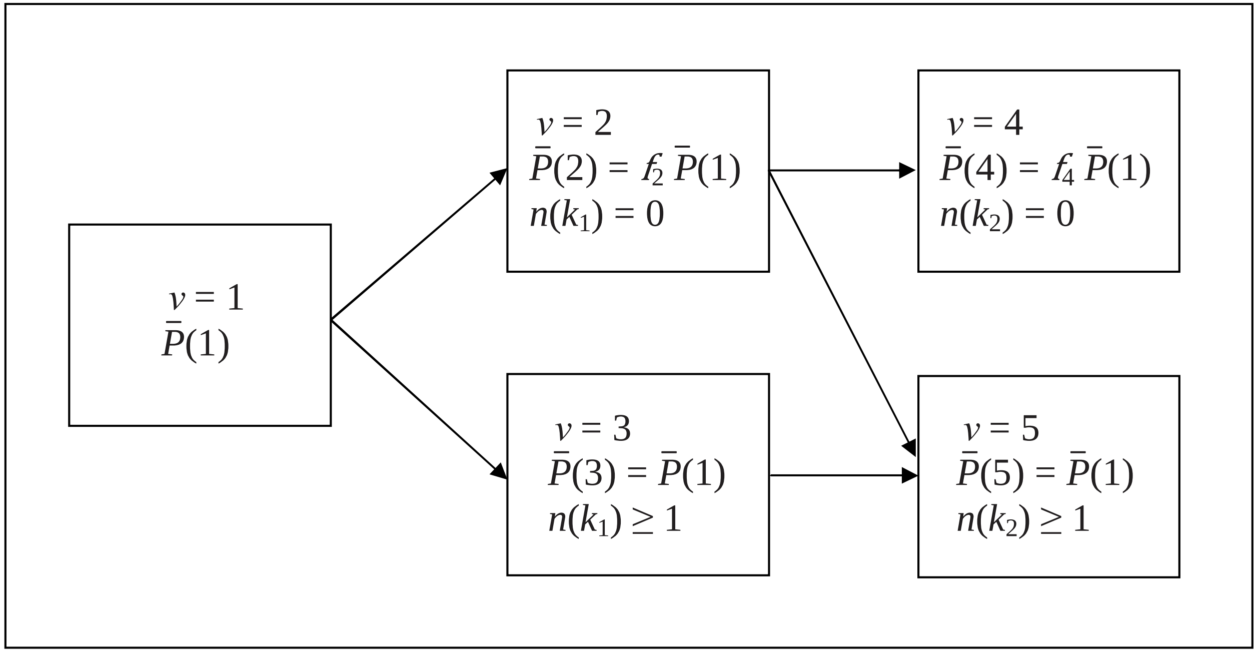


Figure 4 – NCD system: example with 2 adjustment times

BM systems

A *bonus-malus (BM)* system can be defined as a solution of $(^{\circ\circ})$

For example (see Figure 5):

- $r = 1$
- k = time of premium adjustment
- $\nu_{\max} = 5$
- $\bar{P}(1)$ = initial premium
- $\bar{P}(2) = f_2\,\bar{P}(1)$; $\bar{P}(3) = \bar{P}(1)$; $\bar{P}(4) = f_4\,\bar{P}(1)$; $\bar{P}(5) = f_5\,\bar{P}(1)$
- $0 < f_2 < 1 < f_4 < f_5$
- $\sigma(2) = \{0\}$; $\sigma(3) = \{1\}$; $\sigma(4) = \{2\}$; $\sigma(5) = \{3, 4, \ldots\}$

Some particular rating systems (cont'd)

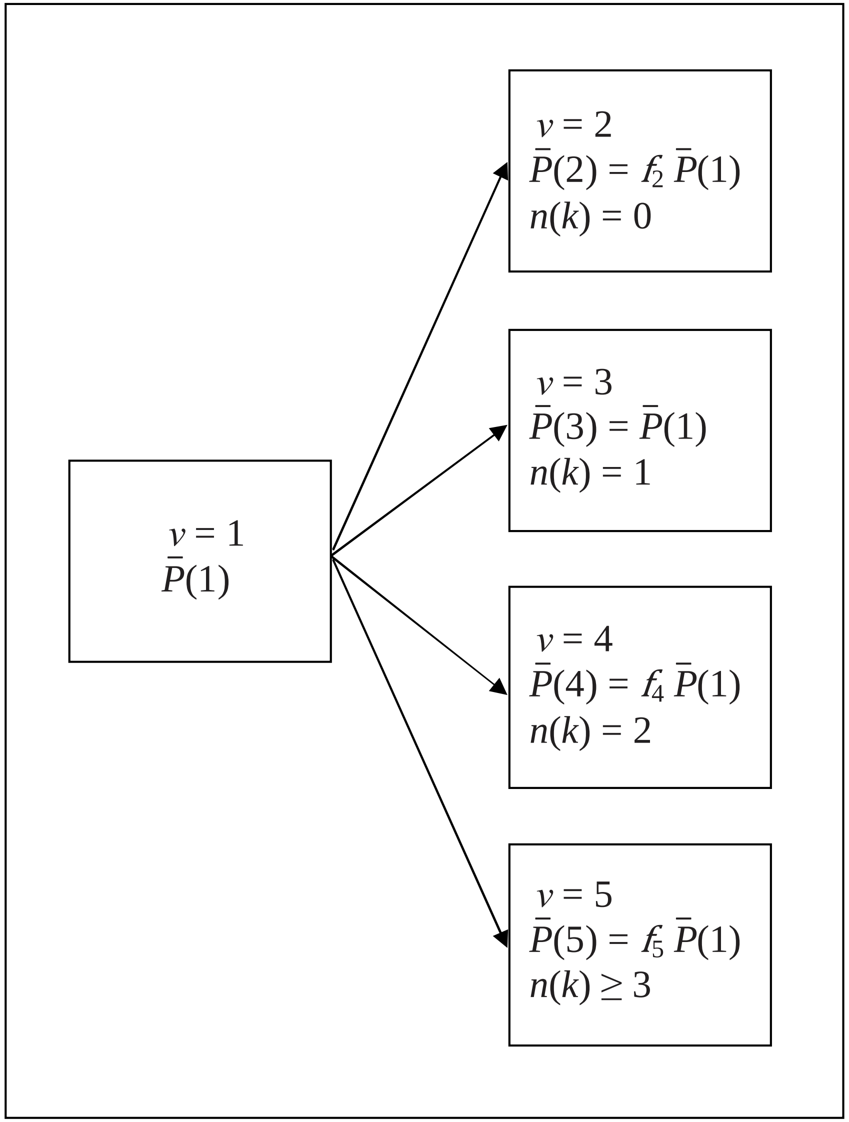


Figure 5 – BM system: an example

AD systems

An *advance-discount (AD)* system can be defined as a solution of $(^{\circ\circ})$

For example (see Figure 6):

- $r = 1$
- k = time of premium adjustment
- $\nu_{\max} = 3$
- $\bar{P}$ = reference premium
- $\bar{P}(1) = \bar{P}(2) = f\,\bar{P};\quad \bar{P}(3) = g\,\bar{P}$
- $f < g$

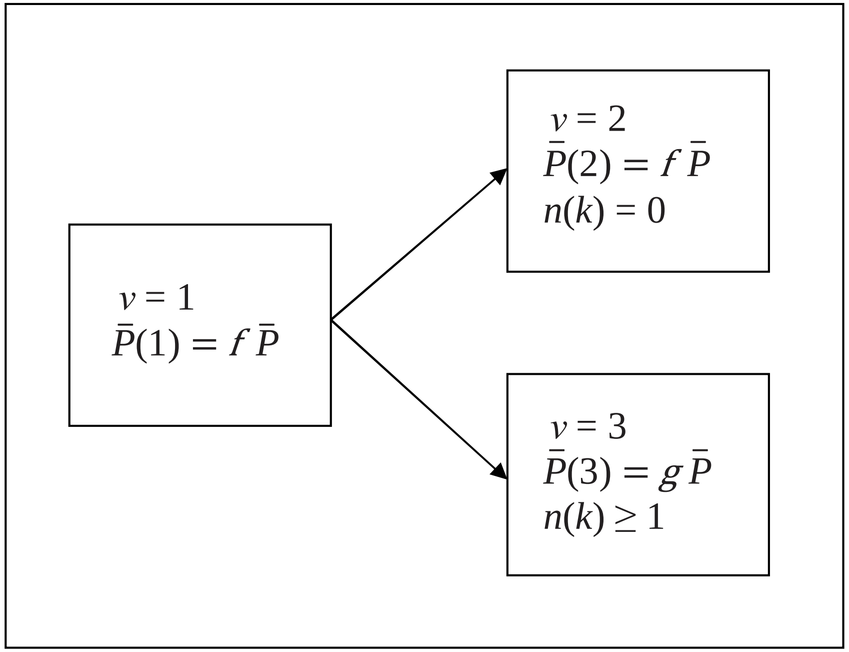


Figure 6 – AD system: example with 1 adjustment time

Another example (see Figure 7):

- $r = 2$
- k_1, k_2 = times of premium adjustment
- $\nu_{\max} = 5$
- $\bar{P}$ = reference premium
- $\bar{P}(1) = f_1\,\bar{P};\quad \bar{P}(2) = f_2\,\bar{P};\quad \bar{P}(3) = f_3\,\bar{P};\quad \bar{P}(4) = f_4\,\bar{P};\quad \bar{P}(5) = f_5\,\bar{P}$
- $f_4 \le f_2 = f_1 < f_3 = f_5$
- $\sigma(2) = \{0\};\quad \sigma(3) = \{1, 2, \dots\};\quad \sigma(4) = \{0\};\quad \sigma(5) = \{1, 2, \dots\}$

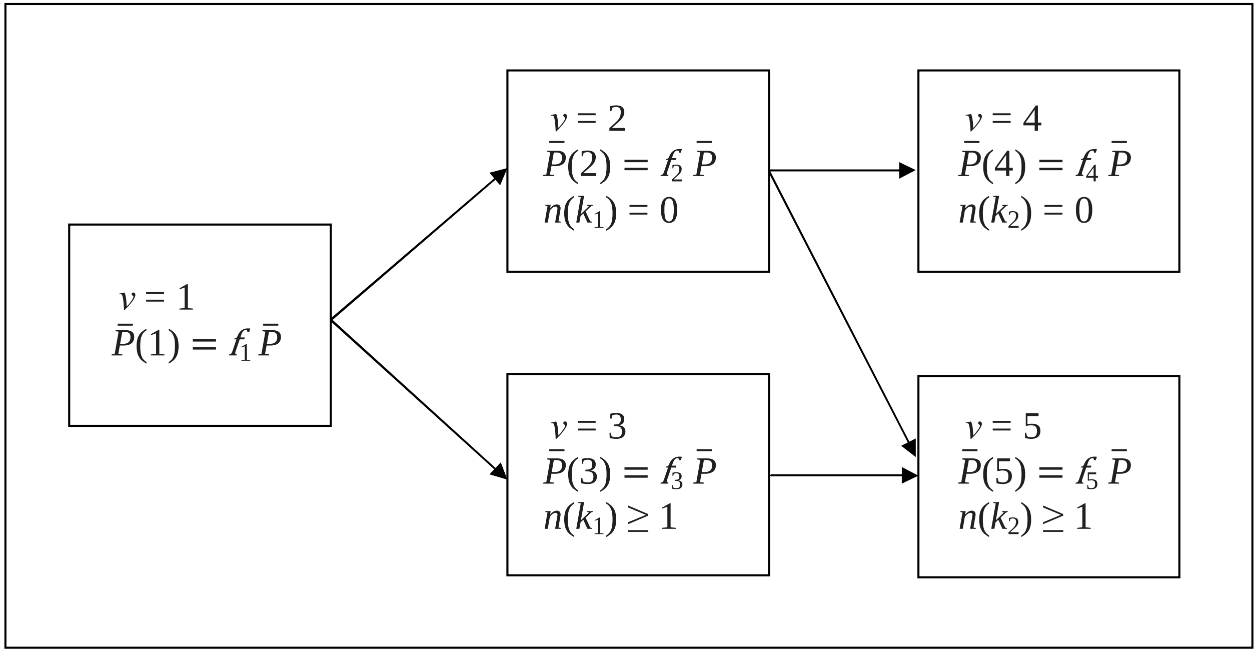


Figure 7 – AD system: example with 2 adjustment times

5.5 NUMERICAL EXAMPLES

The following examples are based on:

$$\mathbb{E}[N_y] = 0.034761 \times 1.032044^y \quad (y \geq 20)$$

Ageing coefficients t_y given by the previous table

Let $x = 40$ = age at policy issue

Parameters of the gamma distribution of Θ:

$$\alpha = 1.1; \quad \beta = 16.83977$$

The following arrangements are considered:

- ▷ straight experience rating (Examples 1, 2, 3, 4)
- ▷ NCD (Examples 5, 6, 7)
- ▷ BM (Example 8)
- ▷ AD (Examples 9, 10, 11)

Example 1

Straight experience rating
$m = 5$
$k = 2$
(see Figure 1)

time k	observed number of claims $n(k)$	node ν	premium $P(\nu)$	$s(\nu)$ = probability of charging the premium $P(\nu)$
0	$-$	1	0.12225	1
2	0	2	0.10780	0.79867
2	≥ 1	3	0.22920	0.20133

Example 2

Straight experience rating
$m = 5$
$k = 3$
(see Figure 1)

time k	observed number of claims $n(k)$	node ν	premium $P(\nu)$	$s(\nu)$ = probability of charging the premium $P(\nu)$
0	$-$	1	0.12416	1
3	0	2	0.09987	0.72142
3	≥ 1	3	0.22377	0.27858

Example 3

Straight experience rating

$m = 5$

$k = 3$

time k	observed number of claims $n(k)$	node ν	premium $P(\nu)$	$s(\nu)$ = probability of charging the premium $P(\nu)$
0	$-$	1	0.12416	1
3	0	2	0.09987	0.72142
3	1	3	0.19066	0.20382
3	2	4	0.28145	0.05497
3	≥ 3	5	0.40456	0.01979

Example 4

Straight experience rating

$m = 10$

$k_1 = 3,\ k_2 = 7$

time k	observed number of claims $n(k)$	node ν	premium $P(\nu)$	$s(\nu)$ = probability of charging the premium $P(\nu)$
0	–	1	0.12416	1
3	0	2	0.10298	0.72142
3	≥ 1	3	0.23075	0.27858
7	0	4	0.08322	0.50517
7	≥ 1	5	0.22792	0.49483

Example 5

NCD system
$m = 5$
$k = 3$
(see Figure 3)

time k	observed number of claims $n(k)$	node ν	premium $\bar{P}(\nu)$	$s(\nu)$ = probability of charging the premium $\bar{P}(\nu)$
0	–	1	0.13532	1
3	0	2	0.10826	0.72142
3	≥ 1	3	0.13532	0.27858

Example 6

NCD system
$m = 5$
$k = 3$
$f_2 = 0.70$
(see Figure 3)

time k	observed number of claims $n(k)$	node ν	premium $\bar{P}(\nu)$	$s(\nu)$ = probability of charging the premium $\bar{P}(\nu)$
0	$-$	1	0.13931	1
3	0	2	0.09751	0.72142
3	≥ 1	3	0.13931	0.27858

Example 7

NCD system
$m = 10$
$k_1 = 3,\ k_2 = 7$
$f_2 = 0.75;\ f_4 = 0.60$
(see Figure 4)

time k	observed number of claims $n(k)$	node ν	premium $\bar{P}(\nu)$	$s(\nu)$ = probability of charging the premium $\bar{P}(\nu)$
0	$-$	1	0.15244	1
3	0	2	0.12195	0.72142
3	≥ 1	3	0.15244	0.27858
7	0	4	0.10671	0.50517
7	≥ 1	5	0.15244	0.49483

Example 8

BM system
$m = 5$
$k = 3$
$f_2 = 0.75;\ f_4 = 1.30;\ f_5 = 1.60$
(see Figure 5)

time k	observed number of claims $n(k)$	node ν	premium $\bar{P}(\nu)$	$s(\nu)$ = probability of charging the premium $\bar{P}(\nu)$
0	–	1	0.13573	1
3	0	2	0.10180	0.72142
3	1	3	0.13573	0.20382
3	2	4	0.17645	0.05497
3	≥ 3	5	0.21717	0.01979

Example 9

AD system
$m = 5$
$k = 2$
$f = 0.90;\ g = 1.20$
(see Figure 6)

time k	observed number of claims $n(k)$	node ν	premium $\bar{P}(\nu)$	$s(\nu)$ = probability of charging the premium $\bar{P}(\nu)$
0	$-$	1	0.12324	1
2	0	2	0.12324	0.79867
2	≥ 1	3	0.16432	0.20133

Example 10

AD system
$m = 5$
$k = 2$
$f = 0.80;\ g = 1.20$
(see Figure 6)

time k	observed number of claims $n(k)$	node ν	premium $\bar{P}(\nu)$	$s(\nu)$ = probability of charging the premium $\bar{P}(\nu)$
0	$-$	1	0.12099	1
2	0	2	0.12099	0.79867
2	≥ 1	3	0.18149	0.20133

Example 11

AD system
$m = 5$
$k = 2$
(see Figure 6)

time k	observed number of claims $n(k)$	node ν	premium $\bar{P}(\nu)$	$s(\nu)$ = probability of charging the premium $\bar{P}(\nu)$
0	$-$	1	0.11500	1
2	0	2	0.11500	0.79867
2	≥ 1	3	0.22727	0.20133

6 THE (AGGREGATE) LONGEVITY RISK IN LIFELONG COVERS

see:

A. Olivieri, E. Pitacco (2002), Premium systems for post-retirement sickness covers, *Belgian Actuarial Bulletin*, 2: 15-25. Available at:
`http://www.belgianactuarialbulletin.be/browse.php?issue=2#2-3`

1. Introduction
2. Sickness insurance and longevity risk
3. Loss functions
4. Premium systems
5. The process risk
6. The uncertainty risk
7. Premium loadings

6.1 INTRODUCTION

Focus on premium systems for lifelong insurance covers providing sickness benefits (viz reimbursement of medical expenses)

Causes of risk affecting lifelong sickness covers:

(a) random number of claim events in any given insured period

(b) random amount (medical expenses refunded) relating to each claim

(c) random lifetime of the insured

Causes (a) and (b):

- ▷ common to all covers in general insurance $\Rightarrow$ safety loading
- ▷ difficulties in lifelong sickness covers because of paucity of data

Cause (c):

- ▷ biometric risk, and in particular longevity risk
- ▷ impact related to the premium system adopted

Premium systems considered in the following;

(1) *single premium* at retirement age, meeting all expected costs

(2) *sequence of level premiums*

(3) *sequence of "natural" premiums*

(4) mixtures of (1) and (2) $\Rightarrow$ *upfront premium* + *sequence of level premiums*

(5) mixtures of (1) and (3) $\Rightarrow$ *upfront premium* + *sequence of premiums proportional to natural premiums*

In particular:

- system (1)
  - ▷ policyholder's point of view: interesting if a lump sum is available at retirement
  - ▷ insurer's point of view: high risk, related to longevity

- system (3)
  - ▷ policyholder's point of view: dramatic increase of premiums at very old ages
  - ▷ insurer's point of view: lowest risk related to longevity
- system (4)
  - ▷ an interesting compromise
  - ▷ adopted by Continuous Care Retirement Communities (CCRC)
    - ◇ advance fee (upfront premium), plus
    - ◇ sequence of periodic fees (periodic premiums), possibly adjusted for inflation

6.2 SICKNESS INSURANCE AND LONGEVITY RISK

Main aspects of mortality trends

(a) decrease in annual probabilities of death

(b) increasing life expectancy

(c) increasing concentration of deaths around the mode of the curve of deaths (rectangularization of the survival curve)

(d) shift of the mode of the curve of deaths towards older ages (expansion)

Need for projected life tables when living benefits are concerned (in particular benefits provided by health insurance products)

Whatever life table is used, future trend is random $\Rightarrow$ risk of *systematic deviations* from expected values

Sickness insurance and longevity risk (cont'd)

Mortality trends at old ages (e.g. beyond age 65)

(a) decrease in annual probabilities of death

(b) increasing life expectancy

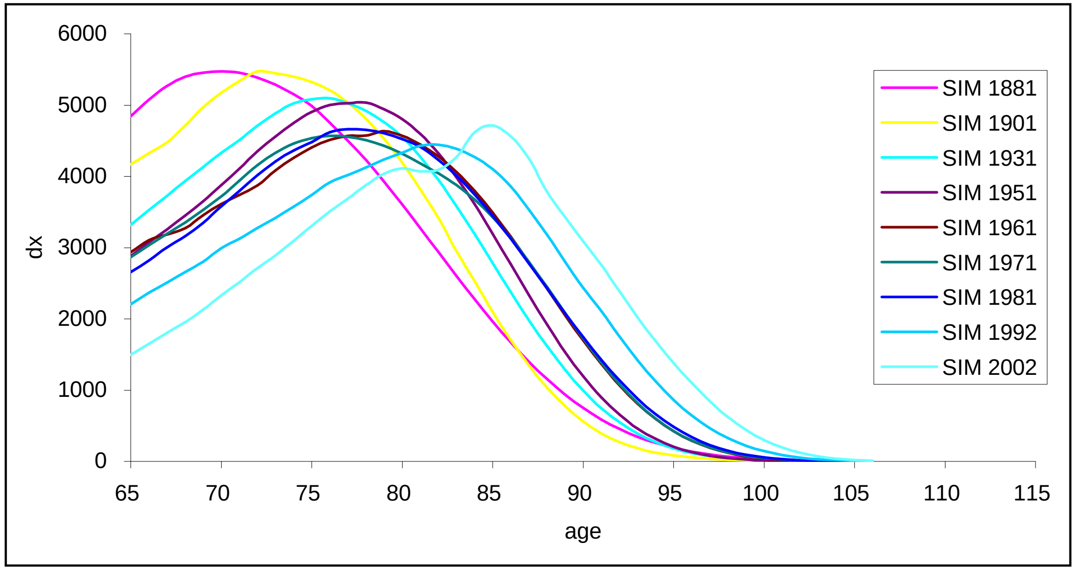
(c) absence of concentration of deaths around the mode of the curve of deaths

(d) shift of the mode of the curve of deaths towards older ages (expansion)

Because of (c) and (d), coexistence of

- random fluctuations around expected values (individual longevity risk)
- systematic deviations from expected values (aggregate longevity risk)

Sickness insurance and longevity risk (cont'd)



Curves of deaths

	SIM 1881	SIM 1901	SIM 1931	SIM 1951	SIM 1961	SIM 1971	SIM 1981	SIM 1992	SIM 2002
$\mathrm{Me}[T_{65}]$	74.45827	75.09749	76.55215	77.42349	78.21735	77.94686	78.27527	80.23987	82.20066
$x_{25}[T_{65}]$	69.80944	70.45377	71.45070	72.16008	72.43802	72.32797	72.65518	73.89806	75.73235
$x_{75}[T_{65}]$	79.95515	80.14873	81.80892	82.63073	83.86049	83.84586	83.96275	86.02055	87.83705
$\mathrm{IQR}[T_{65}]$	10.14570	9.694965	10.35822	10.47065	11.42247	11.51789	11.30757	12.12249	12.10470

Markers of T_{65}

In the context of living benefits, the possibility of facing the (aggregate) longevity risk is strictly related to the type of benefits; in particular

- immediate post-retirement life annuity $\Rightarrow$ single premium $\Rightarrow$ high longevity risk borne by the annuity provider
- post-retirement sickness benefits $\Rightarrow$ possible premium systems including periodic premiums $\Rightarrow$ lower longevity risk borne by the insurer

6.3 LOSS FUNCTIONS

Notation, definitions

- y = insured's age at policy issue (= retirement age)
- N = random number of claims from the time of retirement on
- T_y = future lifetime of the insured
- K_y = curtate future lifetime of the insured
- C_h = random payment for the h-th claim
- $\mathcal{T}_h$ = random time of payment of the h-th claim

Random present value of the payments of the insurer, Y, at the time of retirement (time 0):

$$Y = \sum_{h=1}^{N} C_h \, v^{\mathcal{T}_h}$$

where $v = \frac{1}{1+i}$ = discount factor, i = interest rate

Random present value, Y_{k+1}, at time k of payments in year $(k+1)$-th:

$$Y_{k+1} = \sum_{h:k \le \mathcal{T}_h < k+1} C_h \, v^{\mathcal{T}_h - k}$$

Hence

$$Y = \sum_{k=0}^{K_y} Y_{k+1} \, v^k$$

$\Rightarrow$ link between Y and K_y (or T_y) appears

Assume:

- ▷ claims are uniformly distributed over each year
- ▷ number of claims and claim costs are independent
- ▷ claim costs are equally distributed

Let

- ϕ_{y+k} = expected number of claims in year $(k, k+1)$
- c_{y+k} = expected payment for each claim in the same year

Under the assumptions, the expected present value at time k of payments in year $(k+1)$-th is:

$$\mathbb{E}[Y_{k+1}] = c_{y+k}\,\phi_{y+k}$$

or

$$\mathbb{E}[Y_{k+1}] = c_{y+k}\,\phi_{y+k}\,v^{1/2}$$

The natural premium is of course

$$P_k^{[\mathrm{N}]} = \mathbb{E}[Y_{k+1}]$$

Loss function definition

Let X = random present value at time 0 of premiums

Loss function:

$$L = Y - X$$

or

$$L = \sum_{k=0}^{K_y} Y_{k+1}\, v^k - X \qquad (*)$$

Random items in $(*)$:

- future lifetime
- random number of claims
- costs of claims

In the following $\Rightarrow$ main interest in consequences of the longevity risk
$\Rightarrow$ instead of $(*)$, we adopt the following definition:

$$L = \sum_{k=0}^{K_y} \mathbb{E}[Y_{k+1}]\, v^k - X = \sum_{k=0}^{K_y} P_k^{[\mathrm{N}]}\, v^k - X$$

Mortality assumption

Assume for the random variable T_0 the Weibull distribution, with mortality intensity

$$\mu(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} \quad (a, b > 0)$$

Survival function:

$$S(x) = \mathbb{P}[T_0 > x] = \mathrm{e}^{-(x/a)^b}$$

Density function (“curve of deaths"):

$$f_0(x) = -\frac{dS(x)}{dx} = S(x)\,\mu(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-(x/a)^b}$$

Mode (Lexis point):

$$\xi = a\left(\frac{b-1}{b}\right)^{1/b}$$

Expected lifetime:

$$\mathbb{E}[T_0] = a\,\Gamma\left(\frac{1}{b}+1\right)$$

Variance:

$$\mathbb{V}\mathrm{ar}[T_0] = a^2\left(\Gamma\left(\frac{2}{b}+1\right) - \left(\Gamma\left(\frac{1}{b}+1\right)\right)^2\right)$$

where Γ denotes the complete Gamma function

6.4 PREMIUM SYSTEMS

Whatever the premium system, we adopt the *equivalence principle*, i.e.

$$\mathbb{E}[L] = 0$$

hence

$$\mathbb{E}[X] = \mathbb{E}[Y]$$

Loss function depends on the premium system

- Single premium Π

$$L = \sum_{k=0}^{K_y} P_k^{[\mathrm{N}]} v^k - \Pi$$

- Lifelong annual premiums, π_k paid at time k ($k = 0, 1, \ldots$); we have

$$X = \sum_{k=0}^{K_y} \pi_k \, v^k$$

and then

$$L = \sum_{k=0}^{K_y} (P_k^{[\mathrm{N}]} - \pi_k) \, v^k$$

$$L = \sum_{k=0}^{K_y} P_k^{[\mathrm{N}]} \, v^k - \mathit{\Pi}$$

Premiums π_k for $k = 0, 1, \ldots$ can be, for example

- ▷ level premiums: $\pi_k = \pi$
- ▷ natural premiums: $\pi_k = P_k^{[\mathrm{N}]}$
- ▷ ...

- Mixtures of up-front premium and annual premiums; then

$$X = \mathit{\Pi} + \sum_{k=0}^{K_y} \pi_k \, v^k$$

Let

$$\mathit{\Pi} = \alpha \, \mathbb{E}[Y]; \quad 0 \le \alpha \le 1$$

Equivalence principle fulfilled if

$$\sum_{k=0}^{K_y} \pi_k \, v^k = (1 - \alpha) \, \mathbb{E}[Y]$$

We denote premiums with $\mathit{\Pi}(\alpha)$ and $\pi_k(\alpha)$ for $k = 0, 1, \dots$
In particular:

- ▷ $\alpha = 1 \;\Rightarrow$ single premium $\Rightarrow \; \mathit{\Pi}(1) = \mathit{\Pi}$
- ▷ $\alpha = 0 \;\Rightarrow$ premiums $\pi_k(\alpha)$, $k = 0, 1, \dots \;\Rightarrow\; \mathit{\Pi}(0) = 0$

Loss function:

$$L(\alpha) = \sum_{k=0}^{K_y} \left(P_k^{[\mathrm{N}]} - \pi_k(\alpha) \right) v^k - \mathit{\Pi}(\alpha)$$

Note that $L(\alpha)$ represents the loss function in the general case, $0 \leq \alpha \leq 1$

6.5 THE PROCESS RISK

Portfolio valuations: moments of the loss function

For a given survival function $S(x)$ and related probability of death q, the expected value is:

$$\mathbb{E}[L(\alpha)|S] = \sum_{t=1}^{+\infty} \left[{}_{t-1|1}q_y \left(\sum_{k=0}^{t-1} \left(P_k^{[\mathrm{N}]} - \pi_k(\alpha) \right) v^k - \mathit{\Pi}(\alpha) \right) \right]$$

Note that, if $S(x)$ is also adopted for premium calculation, the equivalence principle implies

$$\mathbb{E}[L(\alpha)|S] = 0$$

Variance:

$$\mathbb{V}\mathrm{ar}[L(\alpha)|S] = \sum_{t=1}^{+\infty} \left[{}_{t-1|1}q_y \left(\sum_{k=0}^{t-1} \left(P_k^{[\mathrm{N}]} - \pi_k(\alpha) \right) v^k - \mathit{\Pi}(\alpha) \right)^2 \right] - \mathbb{E}[L(\alpha)|S]^2$$

Let $S'(x)$ = survival function used to calculate premiums (can in particular coincide with $S'(x)$)

Focus on two premium systems

- Upfront premium + annual premiums proportional to natural premiums; α = quota pertaining to the upfront premium; then:

$$\Pi(\alpha) = \alpha\,\mathbb{E}[Y|S']$$

$$\pi_k(\alpha) = (1-\alpha)\,P_k^{[\mathrm{N}]}; \quad k = 0, 1, \dots$$

Loss function:

$$L_1(\alpha) = \sum_{k=0}^{K_y} \alpha\,P_k^{[\mathrm{N}]}\,v^k - \Pi(\alpha)$$

and then:

$$L_1(\alpha) = \alpha\left(\sum_{k=0}^{K_y} P_k^{[\mathrm{N}]}\,v^k - \mathbb{E}[Y|S']\right)$$

in particular we find:

$$\mathbb{V}\mathrm{ar}[L_1(\alpha)] \propto \alpha^2$$

Note that

- ▷ the variance increases with α, i.e. with the amount of the upfront premium
- ▷ no upfront premium paid ($\alpha = 0$) $\Rightarrow$ $\mathbb{V}\mathrm{ar}[L_1(0]|S) = 0$
  $\Rightarrow$ balance between expected costs and premiums in each year and absence of mortality / longevity risk for the insurer

- Upfront premium + annual level premiums; then:

$$\mathit{\Pi}(\alpha) = \alpha\,\mathbb{E}[Y|S']$$

$$\pi_k(\alpha) = \pi(\alpha); \quad k = 0, 1, \dots$$

Loss function:

$$L_2(\alpha) = \sum_{k=0}^{K_y} \left(P_k^{[\mathrm{N}]} - \pi(\alpha)\right) v^k - \mathit{\Pi}(\alpha)$$

Denote with π the annual premium corresponding to $\alpha = 0$; then

$$\pi(\alpha) = (1 - \alpha)\,\pi$$

and hence

$$L_2(\alpha) = \sum_{k=0}^{K_y} \left(P_k^{[\mathrm{N}]} - \pi\right) v^k - \alpha \left(\mathbb{E}(Y|S') - \pi \sum_{k=0}^{K_y} v^k\right)$$

We find:

$$\mathbb{E}[L_2(\alpha)|S] =$$

$$\sum_{t=1}^{+\infty} \left[{}_{t-1|1}q_y \left(\sum_{k=0}^{t-1} \left(P_k^{[\mathrm{N}]} - \pi\right) v^k - \alpha \left(\mathbb{E}[Y|S'] - \pi \sum_{k=0}^{t-1} v^k \right)\right)\right]$$

$$\mathbb{V}\text{ar}[L_2(\alpha)|S] =$$

$$= \sum_{t=1}^{+\infty} \left[{}_{t-1|1}q_y \left(\sum_{k=0}^{t-1} \left(P_k^{[\text{N}]} - \pi \right) v^k - \alpha \left(\mathbb{E}(Y|S') - \pi \sum_{k=0}^{t-1} v^k \right) \right)^2 \right]$$

$$- \left(\sum_{t=1}^{+\infty} \left[{}_{t-1|1}q_y \left(\sum_{k=0}^{t-1} \left(P_k^{[\text{N}]} - \pi \right) v^k - \alpha \left(\mathbb{E}(Y|S') - \pi \sum_{k=0}^{t-1} v^k \right) \right) \right] \right)^2$$

Moments of the loss function at portfolio level

Loss functions at portfolio level, for a portfolio of (initially) N risks:

$$\mathcal{L}_i(\alpha) = \sum_{j=1}^{N} L_i^{(j)}(\alpha); \quad i = 1, 2$$

where $L_i^{(j)}(\alpha)$ denotes the loss function of the insured j

In a portfolio of N homogeneous and (conditionally) independent risks:

$$\mathbb{E}[\mathcal{L}_i(\alpha)|S] = N\ \mathbb{E}[L_i(\alpha)|S]$$

$$\mathbb{V}\mathrm{ar}[\mathcal{L}_i(\alpha)|S] = N\ \mathbb{V}\mathrm{ar}(L_i(\alpha)|S]$$

Portfolio valuations: riskiness and the portfolio size

Let $\mathcal{Y}$ = random present value of the benefits at portfolio level

Risk index (or *coefficient of variation*):

$$r = \frac{\sigma[\mathcal{Y}|S]}{\mathbb{E}[\mathcal{Y}|S]}$$

For a portfolio of homogeneous and independent risks:

$$\mathbb{E}[\mathcal{Y}|S] = N\,\mathbb{E}[Y|S]$$

$$\mathbb{V}\mathrm{ar}[\mathcal{Y}|S] = N\,\mathbb{V}\mathrm{ar}[Y|S]$$

Hence:

$$r = \frac{1}{\sqrt{N}}\,\frac{\sigma[Y|S]}{\mathbb{E}[Y|S]}$$

$\Rightarrow$ riskiness decreases as the portfolio size increases

Examples

Mortality assumptions:

$$S^{[\mathrm{min}]}(x),\ S^{[\mathrm{med}]}(x),\ S^{[\mathrm{max}]}(x)$$

(see the following table)

Assume:

- age at retirement $y = 65$
- expected number of claims in the year of age $(x, x+1)$

$$\phi_x = 0.1048 \times 0.272859 \times e^{0.029841\,x}$$

- expected cost per claim at age x, $c_x = c = 1$
- rate of interest $i = 0.03$
- mortality assumption for premium calculation $S' = S^{[\mathrm{med}]}$

	$S^{[\min]}(x)$	$S^{[\text{med}]}(x)$	$S^{[\max]}(x)$
a	83.50	85.20	87.00
b	8.00	9.15	10.45
ξ	82.118	84.129	86.167
$\mathbb{E}[T_0]$	78.636	80.742	82.920
$\mathbb{V}\text{ar}[T_0]$	136.120	111.560	91.577

Three projected survival functions

The following tables show:

- variance of the individual loss function conditional on $S^{[\text{med}]}$
- riskiness for portfolio size $N = 100$ and $N = 10\,000$

The process risk (cont'd)

α	$\mathbb{V}\text{ar}[L_1(\alpha)\vert S]$	$\mathbb{V}\text{ar}[L_2(\alpha)\vert S]$
0.0	0.00000	0.14071
0.1	0.02757	0.23755
0.2	0.11029	0.37103
0.3	0.24816	0.54113
0.4	0.44118	0.74785
0.5	0.68935	0.99121
0.6	0.99266	1.27119
0.7	1.35112	1.58780
0.8	1.76473	1.94103
0.9	2.23348	2.33089
1.0	2.75738	2.75738

Variance of the loss function

	$N = 100$			$N = 10\,000$		
	$\mathbb{E}[\mathcal{Y}\|S]$	$\mathbb{V}\mathrm{ar}[\mathcal{Y}\|S]$	$r = \frac{\sigma[\mathcal{Y}\|S]}{\mathbb{E}[\mathcal{Y}\|S]}$	$\mathbb{E}[\mathcal{Y}\|S]$	$\mathbb{V}\mathrm{ar}[\mathcal{Y}\|S]$	$r = \frac{\sigma[\mathcal{Y}\|S]}{\mathbb{E}[\mathcal{Y}\|S]}$
$S^{[\min]}(x)$	337.733	295.406	0.0509	33 773.325	29 540.593	0.0051
$S^{[\mathrm{med}]}(x)$	357.715	275.738	0.0464	35 771.516	27 573.840	0.0046
$S^{[\max]}(x)$	384.815	256.930	0.0417	38 481.540	25 692.981	0.0042

Riskiness for two portfolio sizes

6.6 THE UNCERTAINTY RISK

Portfolio valuations: moments of the loss function

Assign the probabilities

$$\rho^{[\min]},\ \rho^{[\mathrm{med}]},\ \rho^{[\max]}$$

to the survival functions $S^{[\min]}(x)$, $S^{[\mathrm{med}]}(x)$, $S^{[\max]}(x)$ respectively

Unconditional expected value and variance of loss function

$$\mathbb{E}[\mathcal{L}_i(\alpha)] = \mathbb{E}_\rho[\mathbb{E}[\mathcal{L}_i(\alpha)|\mathcal{S})) = N\,\mathbb{E}_\rho[\mathbb{E}[L_i(\alpha)|\mathcal{S}]] = N\,\mathbb{E}[L_i(\alpha)]; \quad i = 1,2$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\mathcal{L}_i(\alpha)] &= \mathbb{E}_\rho[\mathbb{V}\mathrm{ar}[\mathcal{L}_i(\alpha)|\mathcal{S})]] + \mathbb{V}\mathrm{ar}_\rho[\mathbb{E}[\mathcal{L}_i(\alpha)|\mathcal{S}]] \\ &= \underbrace{N\,\mathbb{E}_\rho[\mathbb{V}\mathrm{ar}[L_i(\alpha)|\mathcal{S}]]}_{\text{random fluctuations}} + \underbrace{N^2\,\mathbb{V}\mathrm{ar}_\rho[\mathbb{E}[L_i(\alpha)|\mathcal{S}]]}_{\text{systematic deviations}}; \quad i = 1,2\end{aligned}$$

If $N = \bar{N}$, with

$$\bar{N} = \frac{\mathbb{E}_\rho[\mathbb{V}\mathrm{ar}[L_i(\alpha)|\mathcal{S}]]}{\mathbb{V}\mathrm{ar}_\rho[\mathbb{E}[L_i(\alpha)|\mathcal{S}]]}; \quad i = 1, 2$$

the two terms of the variance are equal

Portfolio valuations: riskiness and the portfolio size

Risk index:

$$r = \frac{\sigma[\mathcal{Y}]}{\mathbb{E}[\mathcal{Y}]} = \left(\underbrace{\frac{1}{N} \frac{\mathbb{E}_\rho[\mathbb{V}\mathrm{ar}[Y|\mathcal{S}]]}{\mathbb{E}^2[Y]}}_{\text{diversifiable}} + \underbrace{\frac{\mathbb{V}\mathrm{ar}_\rho[\mathbb{E}[Y|\mathcal{S}]]}{\mathbb{E}^2[Y]}}_{\text{non-diversifiable}} \right)^{1/2}$$

Examples

Assume

$$\rho^{[\min]} = 0.2,\ \rho^{[\mathrm{med}]} = 0.6,\ \rho^{[\max]} = 0.2$$

Other data as in the previous example

The following tables show:

- ▷ Expected value, variance and relevant components, in the case of premiums proportional to annual expected costs
- ▷ Expected value, variance and relevant components, in the case of level premiums
- ▷ Expected value, variance and risk index as functions of the portfolio size

The uncertainty risk (cont'd)

α	$\mathbb{E}[\mathcal{L}_1(\alpha)]$	$\mathbb{V}\text{ar}[\mathcal{L}_1(\alpha)]$	$\mathbb{E}_\rho\left[\mathbb{V}\text{ar}[\mathcal{L}_1(\alpha)\vert\mathcal{S}]\right]$	$\mathbb{V}\text{ar}_\rho\left[\mathbb{E}[\mathcal{L}_1(\alpha)\vert\mathcal{S}]\right]$	$\bar{N}$
0.0	0.000	0.000	0.000	0.000	–
0.1	14.237	22 747.223	275.910	22 471.313	122.783
0.2	28.473	90 988.893	1 103.641	89 885.252	122.783
0.3	42.710	204 725.009	2 483.192	202 241.817	122.783
0.4	56.947	363 955.571	4 414.563	359 541.008	122.783
0.5	71.183	568 680.580	6 897.755	561 782.826	122.783
0.6	85.420	818 900.036	9 932.767	808 967.269	122.783
0.7	99.657	1 114 613.938	13 519.599	1 101 094.338	122.783
0.8	113.893	1 455 822.286	17 658.252	1 438 164.034	122.783
0.9	128.130	1 842 525.081	22 348.725	1 820 176.355	122.783
1.0	142.367	2 274 722.322	27 591.019	2 247 131.303	122.783

Expected value, variance and relevant components
(premiums proportional to annual expected costs)

The uncertainty risk (cont'd)

α	$\mathbb{E}[\mathcal{L}_2(\alpha)]$	$\mathbb{V}\mathrm{ar}[\mathcal{L}_2(\alpha)]$	$\mathbb{E}_\rho\left[\mathbb{V}\mathrm{ar}[\mathcal{L}_2(\alpha)\vert\mathcal{S}]\right]$	$\mathbb{V}\mathrm{ar}_\rho\left[\mathbb{E}[\mathcal{L}_2(\alpha)\vert\mathcal{S}]\right]$	$\bar{N}$
0.0	42.365	54 388.611	1 430.150	52 958.461	270.051
0.1	52.365	129 488.780	2 407.495	127 081.285	189.445
0.2	62.365	237 240.772	3 749.004	233 491.767	160.563
0.3	72.365	377 644.586	5 454.679	372 189.907	146.556
0.4	82.366	550 700.223	7 524.518	543 175.704	138.528
0.5	92.366	756 407.682	9 958.523	746 449.160	133.412
0.6	102.366	994 766.965	12 756.692	982 010.273	129.904
0.7	112.366	1 265 778.070	15 919.026	1 249 859.044	127.367
0.8	122.366	1 569 440.998	19 445.525	1 549 995.472	125.455
0.9	132.366	1 905 755.748	23 336.190	1 882 419.559	123.969
1.0	142.367	2 274 722.322	27 591.019	2 247 131.303	122.783

Expected value, variance and relevant components (level premiums)

N	$\mathbb{E}[\mathcal{Y}]$	$\mathbb{V}\mathrm{ar}[\mathcal{Y}]$	$\mathbb{E}_\rho\left[\mathbb{V}\mathrm{ar}[\mathcal{Y}\vert\mathcal{S}]\right]$	$\mathbb{V}\mathrm{ar}_\rho\left[\mathbb{E}[\mathcal{Y}\vert\mathcal{S}]\right]$	$r = \frac{\sigma[\mathcal{Y}]}{\mathbb{E}[\mathcal{Y}]}$
100	359.139	500.623	275.910	224.713	0.062
200	718.278	$1\,450.673$	551.820	898.853	0.053
$1\,000$	$3\,591.388$	$25\,230.415$	$2\,759.102$	$22\,471.313$	0.044
$10\,000$	$35\,913.882$	$2\,274\,722.322$	$27\,591.019$	$2\,247\,131.303$	0.042
∞	∞	∞	∞	∞	0.042

Expected value, variance and risk index as functions of the portfolio size

6.7 PREMIUM LOADINGS

Premium loading and loss function

From previous Section: arrangements where annual premiums are proportional to annual expected costs are less risky than systems with level annual premiums

However, level premiums may be preferred

In order to design appealing premium systems, but aiming at limiting risk $\Rightarrow$ level premiums charged with an appropriate safety loading

Let $\pi(\alpha;\lambda)$ = charged premium

Assume a proportional loading:

$$\pi(\alpha;\lambda) = (1+\lambda)\,\pi(\alpha)$$

For this premium arrangement:

- $L_3(\alpha)$ = individual loss function
- $\mathcal{L}_3(\alpha)$ = portfolio loss function

$$L_3(\alpha) = \sum_{k=0}^{K_y} \left(P_k^{[\mathrm{N}]} - \pi(\alpha;\lambda) \right) v^k - \mathit{\Pi}(\alpha)$$

$$\mathcal{L}_3(\alpha) = \sum_{j=1}^{N} L_3^{(j)}(\alpha)$$

Reasonable aims:

$$\mathbb{V}\mathrm{ar}(L_3(\alpha)) = \mathbb{V}\mathrm{ar}(L_1(\alpha)) \qquad (*)$$

$$\mathbb{V}\mathrm{ar}(\mathcal{L}_3(\alpha)) = \mathbb{V}\mathrm{ar}(\mathcal{L}_1(\alpha)) \qquad (**)$$

where $L_1(\alpha)$, $\mathcal{L}_1(\alpha)$ relate to annual premiums proportional to natural premiums

Equations $(*)$, $(**)$ $\Rightarrow$ charge premiums so that the variance of the loss function is the lowest within the probabilistic structure adopted, for a given upfront premium

Process risk

Given the link between the variance of the loss function at individual and portfolio level, loadings resulting from requirements $(*)$ and $(**)$ coincide $\Rightarrow$ focus on the individual case only

It can be proved that, because of the expression of $\mathbb{V}\mathrm{ar}(L_3(\alpha)|S)$, Eq. $(*)$ has the structure

$$A\,\lambda^2 + B\,\lambda + C = 0$$

In the following table:

- $S^{[\mathrm{med}]}$ has been adopted
- when no real solution for equation $(*)$ exists, λ has been set equal to the minimum point λ^* of the function

$$f(\lambda) = \; A\,\lambda^2 + B\,\lambda + C$$

- when the equation is possible, the lower solution has been chosen

Premium loadings (cont'd)

α	λ	λ^*
0.0	0.2144	0.2144
0.1	0.3493	0.3493
0.2	0.3038	0.5180
0.3	0.2727	0.7349
0.4	0.2602	1.0240
0.5	0.2531	1.4288
0.6	0.2486	2.0360
0.7	0.2454	3.0480
0.8	0.2431	5.0721
0.9	0.2413	11.1441
1.0	0.0000	–

Solutions of the loading equation (process risk only)

Uncertainty risk

Eq. $(**)$ must be used

Loading parameter λ depends on the size N of the portfolio

It can be proved that, because of the expression of $\mathbb{V}\mathrm{ar}(\mathcal{L}_3(\alpha))$, Eq. $(**)$ has the structure

$$A(N)\,\lambda^2 + B(N)\,\lambda + C(N) = 0$$

Coefficients $A(N)$, $B(N)$, $C(N)$ are second order polynomials with respect to N

In the following table:

- ▷ when no real solution for equation $(**)$ exists, λ has been set equal to the minimum point λ^* of the function

$$g(\lambda, N) = \; A(N)\,\lambda^2 + B(N)\,\lambda + C(N)$$

- ▷ when the equation is possible, the lower solution has been chosen

As N increases $\Rightarrow$ random fluctuation component tends to vanish
$\Rightarrow$ the required premium loading decreases and has a positive limit

	$N = 1$		$N = 100$		$N = 1\,000$		$N = 10\,000$		$N = 100\,000$	
α	λ	λ^*	λ	λ^*	λ	λ^*	λ	λ^*	λ	λ^*
0.0	0.2157	0.2157	0.1989	0.1989	0.1833	0.1833	0.1800	0.1800	0.1796	0.1796
0.1	0.3508	0.3508	0.3321	0.3321	0.2027	0.3147	0.1844	0.3111	0.1824	0.3107
0.2	0.3085	0.5196	0.2470	0.4986	0.1933	0.4791	0.1823	0.4750	0.1811	0.4745
0.3	0.2764	0.7367	0.2321	0.7127	0.1905	0.6904	0.1817	0.6857	0.1807	0.6852
0.4	0.2635	1.0261	0.2254	0.9982	0.1891	0.9721	0.1814	0.9666	0.1805	0.9660
0.5	0.2564	1.4314	0.2216	1.3978	0.1883	1.3665	0.1812	1.3600	0.1804	1.3592
0.6	0.2517	2.0392	0.2191	1.9973	0.1877	1.9582	0.1810	1.9499	0.1803	1.9490
0.7	0.2485	3.0523	0.2173	2.9964	0.1874	2.9442	0.1809	2.9333	0.1802	2.9320
0.8	0.2461	5.0784	0.2160	4.9946	0.1871	4.9164	0.1809	4.8999	0.1802	4.8981
0.9	0.2443	11.1570	0.2149	10.9890	0.1868	10.8330	0.1808	10.7998	0.1801	10.7961
1.0	0.0000	–	0.0000	–	0.0000	–	0.0000	–	0.0000	–

Solutions of the loading equation (process risk & uncertainty risk)