HEALTH INSURANCE:
ACTUARIAL ASPECTS

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Agenda

1. The need for health-related insurance covers
2. Products in the area of health insurance
3. Between Life and Non-Life insurance: the actuarial structure of sickness insurance
4. Indexation mechanisms
5. Individual experience rating: some models
6. The (aggregate) longevity risk in lifelong covers
1 THE NEED FOR HEALTH-RELATED INSURANCE COVERS

1. Individual flows
2. Aims of health insurance products
3. Risks inherent in the random lifetime
1.1 INDIVIDUAL FLOWS

The following flows are considered

- **inflows:**
  - earned income (wage / salary)
  - pension (+ possible life annuities)

- **outflows: health-related costs**
  - medical expenses (medicines, hospitalization, surgery, etc.)
  - expenses related to long-term care
  - loss of income because of disability (caused by sickness or accident)
Individual flows (cont’d)

Health-related expected costs

Graph showing expected costs over time and age.
Health-related expected costs and their variability
Health-related expected costs and natural premiums (including safety loading)
Individual flows (cont’d)

Income profile

- start of working period
- retirement
Health-related expected costs and whole-life level premiums
Health-related expected costs and temporary level premiums
Individual flows (cont’d)

Health-related expected costs and temporary step-wise level premiums
Individual flows (cont’d)

Level premiums vs natural premiums, and the reserving process
1.2 AIMS OF HEALTH INSURANCE PRODUCTS

1. Replace random costs with deterministic costs (insurance premiums)
   • risk coverage

2. Limit the consequences of time mismatching between income and health costs
   • pre-funding and risk coverage
   • pre-funding $\Rightarrow$ long term products (possibly lifelong)
1.3 RISKS INHERENT IN THE RANDOM LIFETIME

Random lifetime $\Rightarrow$ random duration of

- income (working period and retirement)
- health costs
- premiums

Possible assessment via probability distribution of the lifetime
Risks inherent in the random lifetime (cont’d)

Probability distribution of the random lifetime
Risks inherent in the random lifetime (cont’d)

Probability distributions of the random lifetime  (Source: ISTAT - Italian Males)
Risks inherent in the random lifetime (cont’d)

Difficulties originated by coexistence of:

- random fluctuations of numbers of survivors around expected values
  ⇒ *individual longevity risk*

and, more critical:

- systematic deviations of numbers of survivors from expected values, because of uncertainty in future mortality trend
  ⇒ *aggregate longevity risk*
2 PRODUCTS IN THE AREA OF “HEALTH INSURANCE”

1. General aspects
2. Main products
2.1 GENERAL ASPECTS

“Health insurance”: in several countries, a large set of insurance products providing benefits in the case of need arising from:

- *accident*
- *illness*

and leading to:

- *loss of income* (partial or total, permanent or non-permanent)
- *expenses* (hospitalization, medical and surgery expenses, nursery, etc.)
Area: health insurance belongs to the area of *insurances of the person*, which includes

- *life insurance* (in a strict sense): benefits are due depending on death and survival only, i.e. on the insured’s lifetime

- *health insurance*: benefits are due depending on the health status, and relevant economic consequences (and depending on the lifetime as well)

- *other* insurances of the person: benefits are due depending on events such as marriage, birth of a child, education and professional training of children, etc.

Health insurance (in broad sense) products are usually shared by “life” and “non-life” branches depending on national legislation and regulation.
Health insurance in the context of insurances of the person
2.2 MAIN PRODUCTS

*Types of benefits*

- *Reimbursement benefit*: to meet (totally or partially) health costs, e.g. medical expenses
- *Forfeiture allowance*: amounts stated at policy issue, e.g. to provide an income when the insured is prevented by sickness or injury from working
  - annuity
  - lump sum
- *Service benefit*: care service, e.g. hospital, CCRC (Continuing Care Retirement Communities), etc.
Main products (cont’d)

Classification of products

- Accident insurance
- Sickness insurance
- Health benefits as riders to a basic life insurance cover
- Critical Illness (or Dread Disease) insurance
- Disability annuities
- Long Term Care insurance

Remark

In the following (see products listed in Sect. 3.2) we focus on “sickness insurance”
3 BETWEEN LIFE AND NON-LIFE INSURANCE: THE ACTUARIAL STRUCTURE OF SICKNESS INSURANCE

1. Introduction
2. One-year covers
3. Multi-year covers
4. From the basic model to more general models
3.1 INTRODUCTION

*Life insurance* aspects

mainly concerning medium and long term contracts: disability annuities, LTC insurance, some types of sickness insurance products

- **survival modeling**
  
  benefits are due in case of life ⇒ to be on the “safe side”, survival probabilities should not be underestimated

- **financial issues**
  
  asset accumulation (backing technical reserves), return to policyholders
Non-Life insurance aspects

- claim frequency concerns all types of covers
  - problems: availability, data format, experience monitoring and experience rating
- claim size concerns insurance covers providing reimbursement (e.g. medical expenses), and covers in which benefits depend on some health-related parameter, e.g. the degree of disability
- expenses
  - ascertainment and assessment of claims
  - checking the health status in case of non-necessarily permanent disability
Non-life insurance features: claim frequency, claim severity, ascertainment and assessment of claims, etc.

Life insurance features: life table, interest rate, indexing, etc.

“Life” and “Non-life” aspects in health insurance products
3.2 ONE-YEAR COVERS

Products

1. medical expense reimbursement
2. forfeiture daily allowance for hospitalization
3. forfeiture daily allowance for short-term disability

General features

- Random number $N$ of claims for the generic insured ($N = 0, 1, \ldots$)
- Insurer’s payment: $Y_j$ for the $j$-th claim
- Total annual payment to the generic insured: $S$

$$S = \begin{cases} 
0 & \text{if } N = 0 \\
Y_1 + Y_2 + \cdots + Y_N & \text{if } N > 0
\end{cases}$$
One-year covers (cont’d)

- **Premium calculation**: equivalence principle
- Net premium

\[ \Pi = \mathbb{E}[S] \]

or (to approx take into account timing of payments)

\[ \Pi = \mathbb{E}[S] (1 + i)^{-\frac{1}{2}} \]

where \( i = \) interest rate

- Hypotheses (realistic ?)
  - for any \( N = n \), stochastic independence and identical probability distribution or random variables (r.v.) \( Y_1, Y_2, \ldots, Y_n \)
  - stochastic independence of r.v. \( N, Y_1, Y_2, \ldots \)
  - Hypotheses \( \Rightarrow \) factorizing the expectation of \( S \)

\[ \mathbb{E}[S] = \mathbb{E}[Y] \mathbb{E}[N] \]

with \( Y \) random variable distributed as the \( Y_j \)'s
One-year covers (cont’d)

**Statistical estimation**

- Estimate the quantities $\mathbb{E}[Y], \mathbb{E}[N]$ (technical basis)
- Assumption: “analogous” risks, in terms of amounts (maximum amounts) and exposure time
- Portfolio of *medical expense reimbursement policies*
  - data
    - $r = \text{number of insured risks}$
    - $m = \text{number of claims in the portfolio}$
    - $y_1, y_2, \ldots, y_m = \text{amounts paid}$
  - average claim amount per claim
    $$\bar{y} = \frac{y_1 + y_2 + \cdots + y_m}{m}$$
  - average number of claims per policy (“claim frequency” index)
    $$\phi = \frac{m}{r}$$
One-year covers (cont’d)

- estimates: $\phi \to \mathbb{E}[N]$, $\bar{y} \to \mathbb{E}[Y]$
- premium

$$\Pi = \bar{y} \phi (1 + i)^{-\frac{1}{2}}$$

- Portfolio of *forfeiture daily allowance policies*
  - data
    - $r = \text{number of insured risks}$
    - $m = \text{number of claims in the portfolio}$
    - $g_1, g_2, \ldots, g_m = \text{claim lengths in days}$
  - average length per claim
    $$\bar{g} = \frac{g_1 + g_2 + \cdots + g_m}{m}$$
  - average number of claims per policy ("claim frequency" index)
    $$\phi = \frac{m}{r}$$
One-year covers (cont’d)

- estimates: \( \phi \rightarrow E[N], \bar{g} \rightarrow E[Y] \) (for a unitary daily allowance)
- premium (for a daily allowance \( d \))

\[
\Pi = d \bar{g} \phi (1 + i)^{-1/2}
\]

- morbidity coefficient = average length of claim per policy

\[
\bar{g} \phi = \frac{g_1 + g_2 + \cdots + g_m}{r}
\]

- A more general (and realistic) setting \( \Rightarrow \) allowing for:
  - amounts exposed to risk (annual maximum amounts)
  - exposure time (within 1 observation year)
Risk factors

Split a population into risk classes, according to values assumed by risk factors

Risk factors

- objective: physical characteristics of the insured (age, gender, health records, occupation)
- subjective: personal attitude towards health, which determines the individual demand for medical treatments and, consequently, the application for insurance benefits

Incidence of age: see the following Table
### Example

<table>
<thead>
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<th>$x$</th>
<th>$100 \phi_x$</th>
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100 $\phi = 10.48$

**Average number of claims**

*as a function of the age; males (Source: ISTAT)*

$\phi = \text{overall average}$
One-year covers (cont’d)

### Premiums

- Age as a risk factor ⇒ probability distribution of the random variable $S$ depending on age
- In particular: estimated values $\bar{y}_x, \phi_x, \bar{g}_x$ as functions of age $x$
- Premiums

\[
\Pi_x = \bar{y}_x \phi_x (1 + i)^{-\frac{1}{2}}
\]

\[
\Pi_x = d \bar{g}_x \phi_x (1 + i)^{-\frac{1}{2}}
\]

or, considering just the average number of claims as a function of the age

\[
\Pi_x = \bar{y} \phi_x (1 + i)^{-\frac{1}{2}}
\]

\[
\Pi_x = d \bar{g} \phi_x (1 + i)^{-\frac{1}{2}}
\]
• “Multiplicative” model
  ▷ Assume

\[
\begin{align*}
\phi_x &= \phi t_x \\
\bar{y}_x &= \bar{y} u_x \\
\bar{g}_x &= \bar{g} v_x
\end{align*}
\]

where
- quantities \( \phi, \bar{y}, \bar{g} \) do not depend on age
- coefficients \( t_x, u_x, v_x \) express the age effect (aging coefficients)

▷ Practical interest: assuming that the specific age effect does not change throughout time, claim monitoring can be restricted to quantities \( \phi, \bar{y}, \bar{g} \) observed over the whole portfolio \( \Rightarrow \) more reliable estimates
Example

Forfeiture daily allowance ($d = 100$)

Assumptions (ISTAT data, graduated by ANIA):

$$\phi_x = \frac{0.1048}{\phi} \times \frac{0.272859}{t_x} \times e^{0.029841x}$$

$$\bar{g}_x = \frac{10.91}{\bar{g}} \times \frac{0.655419}{v_x} \times e^{0.008796x}$$
### One-year covers (cont’d)

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**Average number of claims, average time (days) per claim, equivalence premium**
3.3 MULTI-YEARS COVERS

**Premiums**

Medical expense reimbursement or forfeiture daily allowance  
Age \(x\) at policy issue, term \(m\) years  
Single premium  

\[
P_{x,m} = \sum_{h=0}^{m-1} h p_x (1 + i)^{-h} P_{x+h} 
\]

with \(h p_x\) probability, for a person age \(x\), of being alive at age \(x + h\)  
Natural premiums: \(P_x, P_{x+1}, \ldots, P_{x+m-1}\), with  

\[
P_x < P_{x+1} < \cdots < P_{x+m-1} 
\]

(see table above)
Single premium in a multiplicative model

For example, if

$$\Pi_x = \bar{y}_x \phi_x (1 + i)^{-\frac{1}{2}} = \bar{y} \phi u_x t_x (1 + i)^{-\frac{1}{2}}$$

then

$$\Pi_{x,m} = \sum_{h=0}^{m-1} h \rho_x (1 + i)^{-h} \bar{y}_{x+h} \phi_{x+h} (1 + i)^{-\frac{1}{2}}$$

$$= \bar{y} \phi K (\text{indep. of age}) \sum_{h=0}^{m-1} h \rho_x (1 + i)^{-h-\frac{1}{2}} u_{x+h} t_{x+h} w_{x,h} (\text{dependent on age})$$

$$= K \sum_{h=0}^{m-1} w_{x,h}$$

$$= K \pi_{x,m}$$
Multi-year covers (cont’d)

Annual level premium (payable for $m$ years)

$$P_{x,m} = \frac{\Pi_{x,m}}{\ddot{a}_{x:m}}$$

we have

$$P_{x,m} = \sum_{h=0}^{m-1} h p_x (1 + i)^{-h} \Pi_{x+h}$$

thus: annual level premium = arithmetic weighted average of the natural premiums

Consequence: mathematical reserve

Annual level premiums vs natural premiums, and mathematical reserve
Multi-year covers (cont’d)

Example

Hospitalization daily benefit

Data: SIM1992; \( i = 0.03 \); \( d = 100 \); \( \phi_x, \bar{g}_x \) as above

<table>
<thead>
<tr>
<th>( x )</th>
<th>( m = 5 )</th>
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<td>65</td>
<td>1 203.975</td>
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Single premiums
**Multi-year covers (cont’d)**

<table>
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<th>$x$</th>
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<td>65</td>
<td>267.469</td>
<td>–</td>
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*Annual level premiums*
Multi-year covers (cont’d)

Natural premiums and annual level premiums; $x = 45$, $m = 15$

Natural premiums for various ages at policy issue; $m = 15$
Multi-year covers (cont’d)

Reserves

Prospective mathematical reserve (or aging reserve, or senescence reserve)

\[ V_t = \Pi_{x+t,m-t} - P_{x,m} \ddot{a}_{x+t:m-t}; \quad t = 0, 1, \ldots, m \quad (\ast) \]

with

\[ V_0 = V_m = 0 \]

From (\ast) we find

\[ V_t = \Pi_{x+t,1} - P_{x,m} + 1p_{x+t} (1+i)^{-1} (\Pi_{x+t+1,m-t-1} - P_{x,m} \ddot{a}_{x+t+1:m-t-1}) \]

and, as \( \Pi_{x+t,1} = \Pi_{x+t} \), we have the recursion

\[ V_t + P_{x,m} = \Pi_{x+t} + 1p_{x+t} (1 + i)^{-1} V_{t+1} \]

⇒ technical balance in year \((t, t + 1)\)
Example

Hospitalization daily benefit. Data: as above

Reserves for two ages at policy issue; $m = 15$

Reserves for various policy terms; $x = 35$
3.4 FROM THE BASIC MODEL TO MORE GENERAL MODELS

Basic model: a “static” approach, under

- an individual perspective
- a portfolio (or population) perspective

Possible generalizations, in particular allowing for dynamic features:

- claim frequency and claim cost dynamics at portfolio level
- individual claim experience
- longevity dynamics and related consequences in lifelong sickness covers
From the basic model to more general models (cont’d)

Basic model: a “static” approach

Changes in:
- overall claim frequency
- overall claim costs
  → Indexation mechanisms
  [Chapter 4]

Individual experience rating
  → Premium adjustments
  [Chapter 5]

(Aggregate) longevity risk
  In lifelong covers
  → Comparing premium arrangements
  [Chapter 6]

Introducing dynamic aspects
4 INDEXATION MECHANISMS

1. Introduction
2. The adjustment model
4.1 INTRODUCTION

Refer, for example, to medical reimbursement policies
Possible changes, at a portfolio level (or population level), in
- claim frequency
- average cost per claim (e.g. because of inflation)
throughout the policy duration

Approaches:
1. change policy conditions, so that the actuarial value of future benefits keeps constant throughout time; in particular
   (a) raise the deductible (if any)
   (b) lower the maximum amount
2. allow for variations in actuarial values of benefits because of change in claim frequency and/or average cost per claim
   ⇒ indexing policy elements (future premiums and/or reserve) to keep the equivalence principle fulfilled
In what follows, we focus on approach 2 (assuming increase in the actuarial value of benefits)

Refer, for example, to hospitalization benefits
Interest in keeping constant the purchasing power of the daily allowance; then

⇒ indexation of benefits
⇒ need for approach 2
4.2 THE ADJUSTMENT MODEL

Actuarial model

- equivalence at time \( t \) (see the definition of the reserve (*)�)

\[
V_t + P_{x,m} \ddot{a}_{x+t:m-t} = \Pi_{x+t,m-t}
\]

- assume the multiplicative model

\[
\Pi_{x+t,m-t} = K \pi_{x+t,m-t}
\]

- assume that changes only concern the factor \( K \) (whilst do not concern the specific effect of age)

- change in the factor

\[
K \Rightarrow K (1 + j^{[K]})
\]
The adjustment model (cont’d)

• example: medical expense reimbursement

\[ K = \bar{y} \phi \]

▷ change in the average cost per claim because of inflation

\[ K = \bar{y} \phi \implies K (1 + j^{[K]}) = \bar{y} (1 + j^{[K]}) \phi \]

• example: hospitalization benefit (daily allowance)

\[ K = d \bar{g} \phi \]

▷ change in the daily allowance to keep the purchasing power

\[ K = d \bar{g} \phi \implies K (1 + j^{[K]}) = d (1 + j^{[K]}) \bar{g} \phi \]
The adjustment model (cont’d)

• change in the actuarial value

\[ \Pi_{x+t,m-t} \Rightarrow \Pi_{x+t,m-t} (1 + j^{[K]}) = K (1 + j^{[K]}) \pi_{x+t,m-t} \]

• new equivalence condition at time \( t \):

\[ (V_t + P_{x,m} \ddot{a}_{x+t:m-t}) (1 + j^{[K]}) = \Pi_{x+t,m-t} (1 + j^{[K]}) \] (°)

or, in more general terms:

\[ V_t (1 + j^{[V]}) + P_{x,m} (1 + j^{[P]}) \ddot{a}_{x+t:m-t} = \Pi_{x+t,m-t} (1 + j^{[K]}) \] (°°)

with \( j^{[V]}, j^{[P]} \) fulfilling equation (°)

• equivalence condition on the increments:

\[ V_t j^{[V]} + P_{x,m} j^{[P]} \ddot{a}_{x+t:m-t} = \Pi_{x+t,m-t} j^{[K]} \] (°°°)
The adjustment model (cont’d)

- from \((\circ\circ\circ)\) we find:

\[
 j^{[K]} = \frac{V_t \cdot j^{[V]} + P_{x,m} \cdot j^{[P]} \cdot \bar{a}_{x+t:m-t}}{\Pi_{x+t,m-t}}
\]

and then:

\[
 j^{[K]} = \frac{V_t \cdot j^{[V]} + P_{x,m} \cdot j^{[P]} \cdot \bar{a}_{x+t:m-t}}{V_t + P_{x,m} \cdot \bar{a}_{x+t:m-t}}
\]

⇒ relation among the three adjustment rates: \( j^{[K]} \) is the weighted arithmetic mean of \( j^{[V]} \), \( j^{[P]} \)

- usually, application of \((\circ\circ\circ)\) each year, to express an annual adjustment of the actuarial value of the insured benefits

⇒ adjustment rates at time \( t \):

\[
 j^{[K]}_t, \; j^{[V]}_t, \; j^{[P]}_t
\]
The adjustment model (cont’d)

• in practice:
  ▶ increase in the reserve (rate $j_t^{[V]}$) financed by the insurer (profit participation)
  ▶ increase in premiums (rate $j_t^{[P]}$) paid by the policyholder

• in general:
  ▶ if $j_t^{[V]} < j_t^{[K]} \Rightarrow j_t^{[P]} > j_t^{[K]}
  ▶ if $j_t^{[P]} < j_t^{[K]} \Rightarrow j_t^{[V]} > j_t^{[K]}

(because $j_t^{[K]}$ is a weighted arithmetic mean of $j_t^{[V]}$, $j_t^{[P]}$)

Example

Medical expense reimbursement policy
$x = 50$, $m = 15$
annual level premiums payable for the whole policy duration
The adjustment model (cont’d)

<table>
<thead>
<tr>
<th>$t$</th>
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<th>$j_t^{[V]}$</th>
<th>$j_t^{[P]}$</th>
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</tr>
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<td>3</td>
<td>0.00263</td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.00355</td>
<td>0.05</td>
<td>0</td>
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<td>0.05</td>
<td>0</td>
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<td>0</td>
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<td>0.05</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
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<td>0.05</td>
<td>0</td>
</tr>
<tr>
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<td>0.01142</td>
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<td>0</td>
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<td>0</td>
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<tr>
<td>14</td>
<td>0.01346</td>
<td>0.05</td>
<td>0</td>
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</tbody>
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Table 1 - Benefit adjustment maintained via reserve increment only
The adjustment model (cont’d)

<table>
<thead>
<tr>
<th>$t$</th>
<th>$j_t^{[K]}$</th>
<th>$j_t^{[V]}$</th>
<th>$j_t^{[P]}$</th>
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<tbody>
<tr>
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<td>0.06105</td>
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<tr>
<td>2</td>
<td>0.06</td>
<td>0</td>
<td>0.06204</td>
</tr>
<tr>
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<td>0.06</td>
<td>0</td>
<td>0.06299</td>
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<td>0.06</td>
<td>0</td>
<td>0.06389</td>
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<tr>
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<td>0</td>
<td>0.06473</td>
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<td>0</td>
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<td>0</td>
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</tr>
<tr>
<td>14</td>
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</table>

*Table 2 - Only premium increment to maintain a given benefit adjustment*
The adjustment model (cont’d)

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<th>$j_t^{[V]}$</th>
<th>$j_t^{[P]}$</th>
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</thead>
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<td>0.04</td>
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<td>0.06</td>
<td>0.04</td>
<td>0.06171</td>
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<tr>
<td>14</td>
<td>0.06</td>
<td>0.04</td>
<td>0.06445</td>
</tr>
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</table>

*Table 3 - Premium increment, given the reserve increment, to maintain a chosen benefit adjustment*
Remark

Sickness insurance policies (in particular temporary policies) are not “accumulation” products ⇒ the mathematical reserve is small (see numerical examples in the previous section), provided that the policy duration is not too long.

Then:

- the only increment of the reserve cannot maintain the raise in the actuarial value of future benefits (see Table 1)
- the raise in the actuarial value of future benefits can be financed by a reasonable increment of future premiums only (see Table 2)
5 INDIVIDUAL EXPERIENCE RATING: SOME MODELS

see:

1. Introduction
2. The inference model
3. The experience-rating model
4. Some particular rating systems
5. Numerical examples
5.1 INTRODUCTION

In several countries, many policies provide a one-year cover. The insurer is not obliged to renew the policy. In the case of (too many) claims ⇒ no renewal. What is better: no cover or higher (experience-based) premium?

Ratemaking according to individual characteristics

▷ a-priori classification
  based on observable risk factors (age, current health conditions, profession, gender (?), ...)

▷ experience-based classification
  claim experience providing information, in order to partially “replace” risk characteristics which are unobservable at policy issue
In this chapter we define:

- a Bayesian inference model fitting the particular characteristics of sickness insurance (see Sect. 5.2), which in particular provides a “straight” experience rating model (Sect. 5.3)

- some practical rating systems (see Sect. 5.4), such as Bonus Malus (BM) and No-Claim Discount (NCD), relying on the inference model
5.2 THE INFERENCE MODEL

Notation

- \( x \) = insured's age at policy issue, i.e. time 0
- \( m \) = policy term
- \( N_{x+h} \) = random number of claims between age \( x + h \) and \( x + h + 1 \), \( h = 0, 1, \ldots, m - 1 \)
- \( N_x(k) = \sum_{h=0}^{k-1} N_{x+h} \) = cumulated random number of claims up to time \( k \)
- \( \Theta \) = random parameter in the probabilistic structure of \( N_x, N_{x+1}, \ldots, N_{x+m-1} \)
- \( \theta \) = generic outcome of \( \Theta \)
Hypotheses

- given $\Theta = \theta$, the random numbers $N_x, N_{x+1}, \ldots, N_{x+m-1}$ are independent ($\Rightarrow$ conditional independence)

- the probability distribution of $N_{x+h}$, $h = 0, 1, \ldots, m - 1$, is Poisson with parameter $t_{x+h} \theta$, briefly $\text{Pois}(t_{x+h} \theta)$:

\[ \mathbb{P}[N_{x+h} = n|\Theta = \theta] = e^{-t_{x+h} \theta} \frac{(t_{x+h} \theta)^n}{n!}; \quad n = 0, 1, \ldots \]

then:

\[ \mathbb{E}[N_{x+h}|\Theta = \theta] = t_{x+h} \theta \]

$\Rightarrow t_{x+h}$ expresses the age effect; in practice

\[ t_x < t_{x+1} < t_{x+2} < \ldots \]
The inference model (cont’d)

- the probability distribution of $\Theta$ is Gamma with given (positive) parameters $\alpha, \beta$, briefly $\text{Gamma}(\alpha, \beta) \Rightarrow$ probability density function (pdf) given by

$$g(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta}$$

with

$$\mathbb{E}[\Theta] = \frac{\alpha}{\beta}$$

$$\text{Var}[\Theta] = \frac{\alpha}{\beta^2}$$
The inference model  (cont’d)

Some results

• Unconditional distribution of $N_{x+h}, \ h = 0, 1, \ldots, m - 1$

\[
\Pr[N_{x+h} = n] = \int_0^{+\infty} \Pr[N_{x+h} = n | \Theta = \theta] g(\theta) \, d\theta
\]

\[
= \left( \frac{\beta}{t_{x+h}} \right)^\alpha \frac{\Gamma(\alpha + n)}{\Gamma(\alpha) n! \left( \frac{\beta}{t_{x+h}} + 1 \right)^{\alpha+n}}
\]

that is, a negative binomial:

\[
\text{NegBin} \left( \alpha, \frac{\beta}{t_{x+h}} \right)
\]
The inference model (cont’d)

• Then:

\[
\mathbb{E}[N_{x+h}] = \frac{\alpha}{\beta} = t_{x+h} \mathbb{E}[\Theta]
\]

\[
\text{Var}[N_{x+h}] = \alpha \left( \frac{\beta}{t_{x+h}} + 1 \right) \left( \frac{\beta}{t_{x+h}} \right)^2 \left( \frac{\beta}{t_{x+h}} \right)^2
\]

• Given \( \Theta = \theta \), the probability distribution of \( N_x(k) \) is

\[
\text{Pois} \left( \theta \sum_{h=1}^{k} t_{x+h-1} \right)
\]

Remark

The expression \( \mathbb{E}[N_{x+h}] = t_{x+h} \mathbb{E}[\Theta] \) for the expected value corresponds to \( \phi_{x+h} = t_{x+h} \phi \) used in Chap. 3
The inference model (cont’d)

• Then, the unconditional distribution of $N_x(k)$ is

$$
P[N_x(k) = n] = \int_0^{+\infty} P[N_x(k) = n|\Theta = \theta] g(\theta) \, d\theta$$

$$= \frac{\left(\frac{\beta}{\sum_{h=1}^{k} t_{x+h-1}}\right)^\alpha \Gamma(\alpha + n)}{\Gamma(\alpha) \, n! \left(\frac{\beta}{\sum_{h=1}^{k} t_{x+h}} + 1\right)^{\alpha+n}}; \quad n = 0, 1, \ldots$$

that is,

$$\text{NegBin} \left(\alpha, \frac{\beta}{\sum_{h=1}^{k} t_{x+h-1} + 1}\right)$$
The inference procedure

• Claim record \((k < m)\)

\[n_x, \ n_{x+1}, \ldots, n_{x+k-1}\]

• Posterior distribution of the parameter \(\Theta\):

\[
g(\theta|n_x, n_{x+1}, \ldots, n_{x+k-1})
\propto g(\theta) \mathbb{P}[(N_x = n_x) \land (N_{x+1} = n_{x+1}) \land \cdots \land (N_{x+k-1} = n_{x+k-1})|\Theta = \theta]
\]

\[
\propto e^{-\theta \left(\beta + \sum_{h=0}^{k-1} t_{x+h}\right)} \theta^{\alpha + \sum_{h=0}^{k-1} n_{x+h} - 1}
\]

that is, Gamma \(\left(\alpha + \sum_{h=0}^{k-1} n_{x+h}, \ \beta + \sum_{h=0}^{k-1} t_{x+h}\right)\), with

\[
\mathbb{E}[\Theta|n_x, n_{x+1}, \ldots, n_{x+k-1}] = \frac{\alpha + \sum_{h=0}^{k-1} n_{x+h}}{\beta + \sum_{h=0}^{k-1} t_{x+h}}
\]
The inference model (cont’d)

- Unconditional distribution of $N_{x+j}, j \geq k$, calculated by using $g(\theta | n_x, n_{x+1}, \ldots, n_{x+k-1})$ (instead of $g(\theta)$)

- In particular:

\[
\mathbb{E}[N_{x+j} | n_x, n_{x+1}, \ldots, n_{x+k-1}] = t_{x+j} \frac{\alpha + \sum_{h=0}^{k-1} n_{x+h}}{\beta + \sum_{h=0}^{k-1} t_{x+h}}
\]  

(°)

Remark

- sufficient statistics given by \( \left( \sum_{h=0}^{k-1} t_{x+h}, \sum_{h=0}^{k-1} n_{x+h} \right) \)

- Eq. (°) ⇒ credibility formula

\[
\mathbb{E}[N_{x+j} | n_x, n_{x+1}, \ldots, n_{x+k-1}] =
\]

\[
t_{x+j} \left( \frac{\alpha}{\beta} \frac{\beta}{\beta + \sum_{h=0}^{k-1} t_{x+h}} + \frac{\sum_{h=0}^{k-1} n_{x+h}}{\beta + \sum_{h=0}^{k-1} t_{x+h}} \right)
\]

credibility factor $z_{x,k}$
Example 1

Assume:

\[ \mathbb{E}[N_y] = 0.034761 \times 1.032044^y \quad (y \geq 20) \quad (*) \]

Let \( x = 40 \) = age at policy issue

We find:

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \mathbb{E}[N_{40+h}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.123</td>
</tr>
<tr>
<td>1</td>
<td>0.127</td>
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<td>3</td>
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<tr>
<td>4</td>
<td>0.139</td>
</tr>
<tr>
<td>5</td>
<td>0.144</td>
</tr>
</tbody>
</table>

*Expected number of claims*
Example 2

Purpose: to determine the $t$’s (useful in inference procedures)

We know that

$$E[N_y] = t_y E[\Theta]$$

Assume $y'$ as reference age, and set $t_{y'} = 1$

Then:

$$t_y = \frac{E[N_y]}{E[N_{y'}]}$$

For example, with $y' = 20$ and the assumption (*) we find the following Table
### Ageing parameters

<table>
<thead>
<tr>
<th>$y$</th>
<th>$t_y$</th>
</tr>
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<tbody>
<tr>
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<td>1.171</td>
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<td>35</td>
<td>1.605</td>
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<td>1.879</td>
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<td>4.134</td>
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<tr>
<td>70</td>
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</table>
Example 3

Assume
- parameters of the gamma distribution:

\[ \alpha = 1.1; \quad \beta = 16.83977 \]

- age at policy issue \( x = 40 \)

We find the following credibility factors:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( z_{x,k} )</th>
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<tbody>
<tr>
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<td>4</td>
<td>0.319</td>
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<tr>
<td>5</td>
<td>0.373</td>
</tr>
</tbody>
</table>

Credibility factors
We find the following expected values of $N_{45}$, depending on the previous claim experience

| $h$ | $\mathbb{E}[N_{45} | n_{40}, \ldots, n_{44}]$ |
|-----|------------------------------------------|
| 0   | 0.090                                    |
| 1   | 0.172                                    |
| 2   | 0.254                                    |
| 3   | 0.336                                    |
| 4   | 0.418                                    |
| 5   | 0.500                                    |
| 6   | 0.582                                    |
| ... | ...                                      |

*Expected number of claims according to claim experience*
5.3 THE EXPERIENCE-RATING MODEL

Annual level premium, payable for \( m \) years, if no experience rating is adopted

\[
P = \sum_{h=0}^{m-1} h p_x (1 + i)^{-h} \frac{\Pi_{x+h}}{\bar{a}_{x:m}}
\]

where, for a medical expenses insurance cover:

\[
\Pi_{x+h} = \bar{y} \, \mathbb{E}[N_{x+h}] \, (1 + i)^{-\frac{1}{2}}
\]

Assuming \( \bar{y} = 1 \), we have:

\[
P = \sum_{h=0}^{m-1} h p_x (1 + i)^{-h - \frac{1}{2}} \frac{\mathbb{E}[N_{x+h}]}{\bar{a}_{x:m}}
\]

(in line with an experience rating system based on the observed number of claims)
In presence of experience rating

- in principle: in every year different premiums should be determined and charged according to each individual claim record
- in practice: a too complex premium system would be generated

To obtain an applicable premium system, we have to state:

- times at which premium adjustments may occur
- the number of different premiums at each adjustment time
- relationships between claim experience and adjusted premiums

See following notation and Figure 1
The experience-rating model (cont’d)

**Notation**

- $r =$ number of premium adjustments
- $k_1, \ldots, k_r =$ times of premium adjustments; $k = k_1$ if $r = 1$
- $\nu_{\text{max}} =$ number of premiums in the experience rating system
- $\nu =$ index of premium ($\nu = 1, 2, \ldots, \nu_{\text{max}}$)
- $k(\nu) =$ adjustment time at which premium $\nu$ may be charged
- $\sigma(\nu) =$ a set of outcomes of $N_x(k(\nu))$:
  - $N_x(k(\nu)) \in \sigma(\nu) \iff$ premium $\nu$ will be charged (at time $k(\nu)$)
- $q(x, h, n) = \mathbb{P}[N_x(h) = n] =$ probability of $n$ claims up to time $h$
- $s(\nu) = \sum_{n \in \sigma(\nu)} q(x, k(\nu), n) =$ probability that premium $\nu$ will be charged (at time $k(\nu)$)
- $P(\nu) =$ amount of premium $\nu$
Figure 1 – An experience-based rating system; 1 adjustment time
Figure 2 – An experience-based rating system; 2 adjustment times
The experience-rating model (cont’d)

**Premiums**

\[
P(1) = \frac{\sum_{h=0}^{k_1-1} h p_x (1 + i)^{-h} - \frac{1}{2} \mathbb{E}[N_{x+h}] - \frac{1}{2} \mathbb{E}[N_{x+h} + h]}{\bar{a}_{x:k_1}}
\]

\[
P(\nu) = \frac{\sum_{h=k_j}^{k_j+1-1} h^{k_j} p_x + k_j (1 + i)^{-h-k_j} - \frac{1}{2} \mathbb{E}[N_{x+h} | \bigvee_{n \in \sigma(\nu)} (N_x(k_j) = n)]}{\bar{a}_{x+k_j:k_{j+1}-k_j}}
\]

\[
\nu = 2, \ldots, \nu_{\text{max}}; \quad j = 1, \ldots, r, \quad \text{with} \quad k_{r+1} = m
\]
The experience-rating model (cont’d)

Note that:

- Expected values in (*) calculated before any specific experience; then

\[ \mathbb{E}[N_{x+h}] = t_{x+h} \mathbb{E}[\Theta] = t_{x+h} \frac{\alpha}{\beta} \]

- Conditional expected values in (**) depend on the specific information provided by the adoption of premium \( P(\nu) \), i.e. by the set of outcomes of \( N_x(k_j) \) which imply \( P(\nu) \). We have:

\[
\mathbb{E} \left[ N_{x+h} \mid \bigvee_{n \in \sigma(\nu)} N_x(k_j) = n \right] \\
= \sum_{n \in \sigma(\nu)} \mathbb{E} [N_{x+h} \mid N_x(k_j) = n] \frac{q(x, k_j, n)}{\sum_{n \in \sigma(\nu)} q(x, k_j, n)} \\
= \frac{1}{s(\nu)} \sum_{n \in \sigma(\nu)} \mathbb{E} [N_{x+h} \mid N_x(k_j) = n] q(x, k_j, n)
\]
The experience-rating model (cont’d)

- As
  
  \[ N_x(k_j) = \sum_{i=0}^{k_j-1} N_{x+i} \]
  
  we have (according to (°)):

  \[ \mathbb{E}[N_{x+h} \mid N_x(k_j) = n] = t_{x+h} \frac{\alpha + n}{\beta + \sum_{i=0}^{k_j-1} t_{x+i}} \]

By using the equations above, we can calculate

\[ P(1), \quad P(2), \quad \ldots, \quad P(\nu_{\text{max}}) \]

⇒ experience rating system fully defined
5.4 SOME PARTICULAR RATING SYSTEMS

Let $\Pi_{x,m}$ denote the single premium for a $m$-year insurance cover:

\[
\Pi_{x,m} = \sum_{h=0}^{m-1} h p_x (1 + i)^{-h} \Pi_{x+h} = \sum_{h=0}^{m-1} h p_x (1 + i)^{-h-\frac{1}{2}} \mathbb{E}[N_h]
\]

It can be proved that the set of premiums $P(1), P(2), \ldots, P(\nu_{\text{max}})$ (see (\(\ast\)), (\(\ast\ast\)) in Sect. 5.3) fulfills the equivalence principle, that is

\[
\sum_{\nu=1}^{\nu_{\text{max}}} s(\nu) P(\nu) \ddot{a}_{x+k_j:k_{j+1}-k_j} = \Pi_{x,m}
\]

Now consider the $\nu_{\text{max}}$ amounts

$\bar{P}(1), \bar{P}(2), \ldots, \bar{P}(\nu_{\text{max}})$
We say that the $\bar{P}(\nu)$ are *equivalence premiums* if and only if they fulfill the equivalence principle, i.e.

$$\sum_{\nu=1}^{\nu_{\text{max}}} s(\nu) \bar{P}(\nu) \bar{a}_{x+k_j:k_j+1-k_j} = \Pi_{x,m}$$

(○○)

Note that:

- A particular solution of (○○) is given by $P(1), P(2), \ldots, P(\nu_{\text{max}})$

- Other particular solutions of (○○) can be found by stating specific relationships among the premiums, e.g. in order to smooth the sequences of premiums implied by the various claim records

- For example
  - set
    $$\bar{P}(\nu) = f_{\nu} \bar{P}(1); \quad \nu = 2, 3, \ldots, \nu_{\text{max}}$$
  - solve (○○) with respect to $\bar{P}(1)$
  - for given $f_{\nu}$'s, calculate $\bar{P}(2), \ldots, \bar{P}(\nu_{\text{max}})$
Some particular rating systems (cont’d)

- Alternative approach
  - define \( \bar{P} \) as a reference premium (not necessarily charged to the contract, whatever the node)
  - set
    \[
    \bar{P}(\nu) = f_\nu \bar{P}; \quad \nu = 1, 2, \ldots, \nu_{\text{max}}
    \]
  - solve \((\circ \circ)\) with respect to \( \bar{P} \)
  - for given \( f_\nu \)'s, calculate \( \bar{P}(1), \bar{P}(2), \ldots, \bar{P}(\nu_{\text{max}}) \)

- Any premium system

\( \bar{P}(1), \bar{P}(2), \ldots, \bar{P}(\nu_{\text{max}}) \)

(other than \( P(1), P(2), \ldots, P(\nu_{\text{max}}) \)) implies a solidarity effect among insureds
Some particular rating systems (cont’d)

Remarks

1. Note that, when the approach based on the reference premium is adopted, we may find, because of the choice of the reference premium $\bar{P}$ and the parameters $f$’s,

$$\bar{P}(1) < P(1)$$

where $P(1)$ is the initial premium in a straight experience-rating model.

Then

- the insured is not fully financed throughout the first period, i.e. $(0, k_1)$
- loss in case of lapses
Some particular rating systems  (cont’d)

2. As regards the mathematical reserve:
   (a) in the straight experience rating model, the \( P(\nu) \)'s fulfill the
equivalence principle in each period, i.e. \((0, k_1), (k_1, k_2), \ldots, \)
then
   - a small reserve required in each period because of the
     annual increase in natural premiums
   - reserve \( = 0 \) at times \( k_1, k_2, \ldots \)
   (b) in other experience rating systems, the \( \bar{P}(\nu) \)'s only ensure the
equivalence over the cover period \((0, m)\) considered as a
whole, then
   - a higher reserve may be required in each period
   - reserve \( \neq 0 \) at times \( k_1, k_2, \ldots \)
**NCD systems**

A *no-claim discount (NCD)* system can be defined as a solution of \((\circ\circ)\)

For example (see Figure 3):

- \(r = 1\)
- \(k = \) time of premium adjustment
- \(\nu_{\text{max}} = 3\)
- \(\tilde{P}(1) = \) initial premium
- \(\tilde{P}(2) = f_2 \tilde{P}(1); \quad \tilde{P}(3) = \tilde{P}(1)\)
- \(0 < f_2 < 1\)
- \(\sigma(2) = \{0\}; \quad \sigma(3) = \{1, 2, \ldots\}\)
Some particular rating systems (cont’d)

\[v = 1\]
\[\hat{P}(1)\]

\[v = 2\]
\[\hat{P}(2) = f_2 \hat{P}(1)\]
\[n(k) = 0\]

\[v = 3\]
\[\hat{P}(3) = \hat{P}(1)\]
\[n(k) \geq 1\]

Figure 3 – NCD system: example with 1 adjustment time
Some particular rating systems (cont’d)

Another example (see Figure 4):

- \( r = 2 \)
- \( k_1, k_2 \) = times of premium adjustment
- \( \nu_{\text{max}} = 5 \)
- \( \bar{P}(1) \) = initial premium
- \( \bar{P}(2) = f_2 \bar{P}(1); \quad \bar{P}(3) = \bar{P}(1); \quad \bar{P}(4) = f_4 \bar{P}(1); \quad \bar{P}(5) = \bar{P}(1) \)
- \( 0 < f_4 < f_2 < 1 \)
- \( \sigma(2) = \{0\}; \quad \sigma(3) = \{1, 2, \ldots\}; \quad \sigma(4) = \{0\}; \quad \sigma(5) = \{1, 2, \ldots\} \)
Some particular rating systems (cont’d)

Figure 4 – NCD system: example with 2 adjustment times
Some particular rating systems (cont’d)

**BM systems**

A *bonus-malus (BM)* system can be defined as a solution of \((\circ \circ)\)

For example (see Figure 5):

- \(r = 1\)
- \(k = \text{time of premium adjustment}\)
- \(\nu_{\text{max}} = 5\)
- \(\bar{P}(1) = \text{initial premium}\)
- \(\bar{P}(2) = f_2 \bar{P}(1); \quad \bar{P}(3) = \bar{P}(1); \quad \bar{P}(4) = f_4 \bar{P}(1); \quad \bar{P}(5) = f_5 \bar{P}(1)\)
- \(0 < f_2 < 1 < f_4 < f_5\)
- \(\sigma(2) = \{0\}; \quad \sigma(3) = \{1\}; \quad \sigma(4) = \{2\}; \quad \sigma(5) = \{3, 4, \ldots \}\)
Some particular rating systems (cont’d)

\[ P(1) \]

- \( v = 2 \)
  - \( \bar{P}(2) = f_2 \bar{P}(1) \)
  - \( n(k) = 0 \)

- \( v = 3 \)
  - \( \bar{P}(3) = \bar{P}(1) \)
  - \( n(k) = 1 \)

- \( v = 4 \)
  - \( \bar{P}(4) = f_4 \bar{P}(1) \)
  - \( n(k) = 2 \)

- \( v = 5 \)
  - \( \bar{P}(5) = f_5 \bar{P}(1) \)
  - \( n(k) \geq 3 \)

Figure 5 – BM system: an example
Some particular rating systems (cont’d)

**AD systems**

An *advance-discount (AD)* system can be defined as a solution of $(\circ \circ)$

For example (see Figure 6):

- $r = 1$
- $k =$ time of premium adjustment
- $\nu_{\text{max}} = 3$
- $\bar{P} =$ reference premium
- $\bar{P}(1) = \bar{P}(2) = f \bar{P}; \quad \bar{P}(3) = g \bar{P}$
- $f < g$
Some particular rating systems (cont’d)

\[ v = 1 \]
\[ \bar{P}(1) = f \bar{P} \]

\[ v = 2 \]
\[ \bar{P}(2) = f \bar{P} \]
\[ n(k) = 0 \]

\[ v = 3 \]
\[ \bar{P}(3) = g \bar{P} \]
\[ n(k) \geq 1 \]

Figure 6 – AD system: example with 1 adjustment time
Some particular rating systems (cont’d)

Another example (see Figure 7):

- $r = 2$
- $k_1, k_2 =$ times of premium adjustment
- $\nu_{\text{max}} = 5$
- $\bar{P} =$ reference premium
- $\bar{P}(1) = f_1 \bar{P}; \quad \bar{P}(2) = f_2 \bar{P}; \quad \bar{P}(3) = f_3 \bar{P}; \quad \bar{P}(4) = f_4 \bar{P}; \quad \bar{P}(5) = f_5 \bar{P}$
- $f_4 \leq f_2 = f_1 < f_3 = f_5$
- $\sigma(2) = \{0\}; \quad \sigma(3) = \{1, 2, \ldots\}; \quad \sigma(4) = \{0\}; \quad \sigma(5) = \{1, 2, \ldots\}$
Some particular rating systems (cont’d)

\[
\begin{align*}
\nu = 1 & \quad \bar{P}(1) = f_1 \bar{P} \\
\nu = 2 & \quad \bar{P}(2) = f_2 \bar{P} \\
& \quad n(k_1) = 0 \\
\nu = 3 & \quad \bar{P}(3) = f_3 \bar{P} \\
& \quad n(k_1) \geq 1 \\
\nu = 4 & \quad \bar{P}(4) = f_4 \bar{P} \\
& \quad n(k_2) = 0 \\
\nu = 5 & \quad \bar{P}(5) = f_5 \bar{P} \\
& \quad n(k_2) \geq 1
\end{align*}
\]

Figure 7 – AD system: example with 2 adjustment times
5.5 NUMERICAL EXAMPLES

The following examples are based on:

\[ \mathbb{E}[N_y] = 0.034761 \times 1.032044^y \quad (y \geq 20) \]

Ageing coefficients \( t_y \) given by the previous table
Let \( x = 40 \) = age at policy issue
Parameters of the gamma distribution of \( \Theta \):

\[ \alpha = 1.1; \quad \beta = 16.83977 \]

The following arrangements are considered:

- straight experience rating (Examples 1, 2, 3, 4)
- NCD (Examples 5, 6, 7)
- BM (Example 8)
- AD (Examples 9, 10, 11)
**Example 1**

**Straight experience rating**

\[ m = 5 \]

\[ k = 2 \]

(see Figure 1)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( P(\nu) )</th>
<th>( s(\nu) = \text{probability of charging} ) the premium ( P(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
<td>1</td>
<td>0.12225</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0.10780</td>
<td>0.79867</td>
</tr>
<tr>
<td>2</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.22920</td>
<td>0.20133</td>
</tr>
</tbody>
</table>
Example 2

Straight experience rating

\( m = 5 \)
\( k = 3 \)
(see Figure 1)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( P(\nu) )</th>
<th>( s(\nu) = ) probability of charging the premium ( P(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>−</td>
<td>1</td>
<td>0.12416</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0.09987</td>
<td>0.72142</td>
</tr>
<tr>
<td>3</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.22377</td>
<td>0.27858</td>
</tr>
</tbody>
</table>
Example 3

Straight experience rating

\[ m = 5 \]
\[ k = 3 \]

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( P(\nu) )</th>
<th>( s(\nu) = \text{probability of charging} ) the premium ( P(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-)</td>
<td>1</td>
<td>0.12416</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0.09987</td>
<td>0.72142</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0.19066</td>
<td>0.20382</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>0.28145</td>
<td>0.05497</td>
</tr>
<tr>
<td>3</td>
<td>( \geq 3 )</td>
<td>5</td>
<td>0.40456</td>
<td>0.01979</td>
</tr>
</tbody>
</table>
Example 4

Straight experience rating

\( m = 10 \)

\( k_1 = 3, \ k_2 = 7 \)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( P(\nu) )</th>
<th>( s(\nu) = ) probability of charging the premium ( P(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>−</td>
<td>1</td>
<td>0.12416</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0.10298</td>
<td>0.72142</td>
</tr>
<tr>
<td>3</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.23075</td>
<td>0.27858</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>4</td>
<td>0.08322</td>
<td>0.50517</td>
</tr>
<tr>
<td>7</td>
<td>( \geq 1 )</td>
<td>5</td>
<td>0.22792</td>
<td>0.49483</td>
</tr>
</tbody>
</table>
**Example 5**

NCD system

\[
m = 5 \\
k = 3
\]
(see Figure 3)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( \bar{P}(\nu) )</th>
<th>( s(\nu) = \text{probability of charging the premium } \bar{P}(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
<td>1</td>
<td>0.13532</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0.10826</td>
<td>0.72142</td>
</tr>
<tr>
<td>3</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.13532</td>
<td>0.27858</td>
</tr>
</tbody>
</table>
Example 6

NCD system

\[ m = 5 \]
\[ k = 3 \]
\[ f_2 = 0.70 \]

(see Figure 3)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( \bar{P}(\nu) )</th>
<th>( s(\nu) = \text{probability of charging the premium} \bar{P}(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>−</td>
<td>1</td>
<td>0.13931</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0.09751</td>
<td>0.72142</td>
</tr>
<tr>
<td>3</td>
<td>≥1</td>
<td>3</td>
<td>0.13931</td>
<td>0.27858</td>
</tr>
</tbody>
</table>
**Numerical examples (cont’d)**

**Example 7**

NCD system

\[ m = 10 \]

\[ k_1 = 3, \; k_2 = 7 \]

\[ f_2 = 0.75; \; f_4 = 0.60 \]

(see Figure 4)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( \bar{P}(\nu) )</th>
<th>( s(\nu) = ) probability of charging the premium ( \bar{P}(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
<td>1</td>
<td>0.15244</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0.12195</td>
<td>0.72142</td>
</tr>
<tr>
<td>3</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.15244</td>
<td>0.27858</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>4</td>
<td>0.10671</td>
<td>0.50517</td>
</tr>
<tr>
<td>7</td>
<td>( \geq 1 )</td>
<td>5</td>
<td>0.15244</td>
<td>0.49483</td>
</tr>
</tbody>
</table>
**Example 8**

BM system

\[ m = 5 \]
\[ k = 3 \]
\[ f_2 = 0.75; \quad f_4 = 1.30; \quad f_5 = 1.60 \]

(see Figure 5)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( \bar{P}(\nu) )</th>
<th>( s(\nu) ) = probability of charging the premium ( \bar{P}(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
<td>1</td>
<td>0.13573</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
<td>2</td>
<td>0.10180</td>
<td>0.72142</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0.13573</td>
<td>0.20382</td>
</tr>
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<td>2</td>
<td>4</td>
<td>0.17645</td>
<td>0.05497</td>
</tr>
<tr>
<td>3</td>
<td>( \geq 3 )</td>
<td>5</td>
<td>0.21717</td>
<td>0.01979</td>
</tr>
</tbody>
</table>
Example 9

AD system

\[ m = 5 \]
\[ k = 2 \]
\[ f = 0.90; \quad g = 1.20 \]

(see Figure 6)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( \bar{P}(\nu) )</th>
<th>( s(\nu) = ) probability of charging the premium ( \bar{P}(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
<td>1</td>
<td>0.12324</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0.12324</td>
<td>0.79867</td>
</tr>
<tr>
<td>2</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.16432</td>
<td>0.20133</td>
</tr>
</tbody>
</table>
**Example 10**

AD system

\[ m = 5 \]
\[ k = 2 \]
\[ f = 0.80; \quad g = 1.20 \]
(see Figure 6)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( \bar{P}(\nu) )</th>
<th>( s(\nu) = \text{probability of charging the premium} ) ( \bar{P}(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
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<td>0.12099</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0.12099</td>
<td>0.79867</td>
</tr>
<tr>
<td>2</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.18149</td>
<td>0.20133</td>
</tr>
</tbody>
</table>
Numerical examples (cont’d)

**Example 11**

AD system

\[ m = 5 \]
\[ k = 2 \]
(see Figure 6)

<table>
<thead>
<tr>
<th>time ( k )</th>
<th>observed number of claims ( n(k) )</th>
<th>node ( \nu )</th>
<th>premium ( \bar{P}(\nu) )</th>
<th>( s(\nu) = \text{probability of charging the premium } \bar{P}(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>0.11500</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0.11500</td>
<td>0.79867</td>
</tr>
<tr>
<td>2</td>
<td>( \geq 1 )</td>
<td>3</td>
<td>0.22727</td>
<td>0.20133</td>
</tr>
</tbody>
</table>
6 THE (AGGREGATE) LONGEVITY RISK IN LIFELONG COVERS

see:

1. Introduction
2. Sickness insurance and longevity risk
3. Loss functions
4. Premium systems
5. The process risk
6. The uncertainty risk
7. Premium loadings
6.1 INTRODUCTION

Focus on premium systems for lifelong insurance covers providing sickness benefits (viz reimbursement of medical expenses)

Causes of risk affecting lifelong sickness covers:

(a) random number of claim events in any given insured period
(b) random amount (medical expenses refunded) relating to each claim
(c) random lifetime of the insured

Causes (a) and (b):

- common to all covers in general insurance ⇒ safety loading
- difficulties in lifelong sickness covers because of paucity of data

Cause (c):

- biometric risk, and in particular longevity risk
- impact related to the premium system adopted
Premium systems considered in the following;

(1) *single premium* at retirement age, meeting all expected costs

(2) *sequence of level premiums*

(3) *sequence of “natural” premiums*

(4) mixtures of (1) and (2) ⇒ *upfront premium + sequence of level premiums*

(5) mixtures of (1) and (3) ⇒ *upfront premium + sequence of premiums proportional to natural premiums*

In particular:

- system (1)
  - policyholder’s point of view: interesting if a lump sum is available at retirement
  - insurer’s point of view: high risk, related to longevity
Introduction (cont’d)

- system (3)
  - policyholder’s point of view: dramatic increase of premiums at very old ages
  - insurer’s point of view: lowest risk related to longevity

- system (4)
  - an interesting compromise
  - adopted by Continuous Care Retirement Communities (CCRC)
    - advance fee (upfront premium), plus
    - sequence of periodic fees (periodic premiums), possibly adjusted for inflation
6.2 SICKNESS INSURANCE AND LONGEVITY RISK

Main aspects of mortality trends
(a) decrease in annual probabilities of death
(b) increasing life expectancy
(c) increasing concentration of deaths around the mode of the curve of deaths (rectangularization of the survival curve)
(d) shift of the mode of the curve of deaths towards older ages (expansion)

Need for projected life tables when living benefits are concerned (in particular benefits provided by health insurance products)

Whatever life table is used, future trend is random ⇒ risk of systematic deviations from expected values
Sickness insurance and longevity risk (cont’d)

Mortality trends at old ages (e.g. beyond age 65)
(a) decrease in annual probabilities of death
(b) increasing life expectancy
(c) absence of concentration of deaths around the mode of the curve of deaths
(d) shift of the mode of the curve of deaths towards older ages (expansion)

Because of (c) and (d), coexistence of
- random fluctuations around expected values (individual longevity risk)
- systematic deviations from expected values (aggregate longevity risk)
Sickness insurance and longevity risk (cont’d)

Curves of deaths

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td>Me[$T_{65}$]</td>
<td>74.45827</td>
<td>75.09749</td>
<td>76.55215</td>
<td>77.42349</td>
<td>78.21735</td>
<td>77.94686</td>
<td>78.27527</td>
<td>80.23987</td>
<td>82.20066</td>
</tr>
<tr>
<td>$x_{25}[T_{65}]$</td>
<td>69.80944</td>
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<td>72.32797</td>
<td>72.65518</td>
<td>73.89806</td>
<td>75.73235</td>
</tr>
<tr>
<td>$x_{75}[T_{65}]$</td>
<td>79.95515</td>
<td>80.14873</td>
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<td>83.96275</td>
<td>86.02055</td>
<td>87.83705</td>
</tr>
</tbody>
</table>

Markers of $T_{65}$
In the context of living benefits, the possibility of facing the (aggregate) longevity risk is strictly related to the type of benefits; in particular

- immediate post-retirement life annuity $\implies$ single premium $\implies$ high longevity risk borne by the annuity provider

- post-retirement sickness benefits $\implies$ possible premium systems including periodic premiums $\implies$ lower longevity risk borne by the insurer
6.3 LOSS FUNCTIONS

Notation, definitions

- \( y \) = insured’s age at policy issue (= retirement age)
- \( N \) = random number of claims from the time of retirement on
- \( T_y \) = future lifetime of the insured
- \( K_y \) = curtate future lifetime of the insured
- \( C_h \) = random payment for the \( h \)-th claim
- \( T_h \) = random time of payment of the \( h \)-th claim

Random present value of the payments of the insurer, \( Y \), at the time of retirement (time 0):

\[
Y = \sum_{h=1}^{N} C_h \cdot v^{T_h}
\]

where \( v = \frac{1}{1+i} \) = discount factor, \( i \) = interest rate
Random present value, $Y_{k+1}$, at time $k$ of payments in year $(k + 1)$-th:

$$Y_{k+1} = \sum_{h:k \leq T_h < k+1} C_h \, v^{T_h-k}$$

Hence

$$Y = \sum_{k=0}^{K_y} Y_{k+1} \, v^k$$

$\Rightarrow$ link between $Y$ and $K_y$ (or $T_y$) appears

Assume:

- claims are uniformly distributed over each year
- number of claims and claim costs are independent
- claim costs are equally distributed
Let

- $\phi_{y+k}$ = expected number of claims in year $(k, k + 1)$
- $c_{y+k}$ = expected payment for each claim in the same year

Under the assumptions, the expected present value at time $k$ of payments in year $(k + 1)$-th is:

$$\mathbb{E}[Y_{k+1}] = c_{y+k} \phi_{y+k}$$

or

$$\mathbb{E}[Y_{k+1}] = c_{y+k} \phi_{y+k} v^{1/2}$$

The natural premium is of course

$$P_k^{[N]} = \mathbb{E}[Y_{k+1}]$$
Loss function definition
Let \( X = \) random present value at time 0 of premiums
Loss function:

\[
L = Y - X
\]

or

\[
L = \sum_{k=0}^{K_y} Y_{k+1} v^k - X
\]  \( (*) \)

Random items in \( (*) \):
- future lifetime
- random number of claims
- costs of claims

In the following \( \Rightarrow \) main interest in consequences of the longevity risk
\( \Rightarrow \) instead of \( (*) \), we adopt the following definition:
Loss functions (cont’d)

\[ L = \sum_{k=0}^{K_y} \mathbb{E}[Y_{k+1}] v^k - X = \sum_{k=0}^{K_y} P_k^{[N]} v^k - X \]

**Mortality assumption**

Assume for the random variable \( T_0 \) the Weibull distribution, with mortality intensity

\[ \mu(x) = \frac{b}{a} \left( \frac{x}{a} \right)^{b-1} \quad (a, b > 0) \]

Survival function:

\[ S(x) = \mathbb{P}[T_0 > x] = e^{-(x/a)^b} \]

Density function (“curve of deaths”):

\[ f_0(x) = -\frac{dS(x)}{dx} = S(x) \mu(x) = \frac{b}{a} \left( \frac{x}{a} \right)^{b-1} e^{-(x/a)^b} \]
Mode (Lexis point):

\[ \xi = a \left( \frac{b - 1}{b} \right)^{1/b} \]

Expected lifetime:

\[ \mathbb{E}[T_0] = a \Gamma \left( \frac{1}{b} + 1 \right) \]

Variance:

\[ \text{Var}[T_0] = a^2 \left( \Gamma \left( \frac{2}{b} + 1 \right) - \left( \Gamma \left( \frac{1}{b} + 1 \right) \right)^2 \right) \]

where \( \Gamma \) denotes the complete Gamma function.
Whatever the premium system, we adopt the *equivalence principle*, i.e.

\[ \mathbb{E}[L] = 0 \]

hence

\[ \mathbb{E}[X] = \mathbb{E}[Y] \]

Loss function depends on the premium system

- Single premium \( \Pi \)

\[ L = \sum_{k=0}^{K_y} P_k^{[N]} \nu^k - \Pi \]
Premium systems (cont’d)

- Lifelong annual premiums, $\pi_k$, paid at time $k$ ($k = 0, 1, \ldots$); we have

\[
X = \sum_{k=0}^{K_y} \pi_k v^k
\]

and then

\[
L = \sum_{k=0}^{K_y} (P_k^{[N]} - \pi_k) v^k
\]

\[
L = \sum_{k=0}^{K_y} P_k^{[N]} v^k - \Pi
\]

Premiums $\pi_k$ for $k = 0, 1, \ldots$ can be, for example

- level premiums: $\pi_k = \pi$
- natural premiums: $\pi_k = P_k^{[N]}$
- \ldots
Premium systems (cont’d)

- Mixtures of up-front premium and annual premiums; then

\[ X = \Pi + \sum_{k=0}^{K_y} \pi_k v^k \]

Let

\[ \Pi = \alpha \mathbb{E}[Y]; \quad 0 \leq \alpha \leq 1 \]

Equivalence principle fulfilled if

\[ \sum_{k=0}^{K_y} \pi_k v^k = (1 - \alpha) \mathbb{E}[Y] \]

We denote premiums with \( \Pi(\alpha) \) and \( \pi_k(\alpha) \) for \( k = 0, 1, \ldots \)

In particular:

\[ \alpha = 1 \Rightarrow \text{single premium} \Rightarrow \Pi(1) = \Pi \]

\[ \alpha = 0 \Rightarrow \text{premises} \pi_k(\alpha), \ k = 0, 1, \ldots \Rightarrow \Pi(0) = 0 \]
Loss function:

\[ L(\alpha) = \sum_{k=0}^{K_y} \left( P_k^{[N]} - \pi_k(\alpha) \right) v^k - \Pi(\alpha) \]

Note that \( L(\alpha) \) represents the loss function in the general case, \( 0 \leq \alpha \leq 1 \)
6.5 THE PROCESS RISK

*Portfolio valuations: moments of the loss function*

For a given survival function \( S(x) \) and related probability of death \( q \), the expected value is:

\[
\mathbb{E}[L(\alpha) | S] = \sum_{t=1}^{+\infty} \left( t-1 \right) q_y \left( \sum_{k=0}^{t-1} \left( P_k^{[N]} - \pi_k(\alpha) \right) v^k - \Pi(\alpha) \right)
\]

Note that, if \( S(x) \) is also adopted for premium calculation, the equivalence principle implies

\[
\mathbb{E}[L(\alpha) | S] = 0
\]

Variance:

\[
\text{Var}[L(\alpha) | S] = \sum_{t=1}^{+\infty} \left( t-1 \right) q_y \left( \sum_{k=0}^{t-1} \left( P_k^{[N]} - \pi_k(\alpha) \right) v^k - \Pi(\alpha) \right)^2 - \mathbb{E}[L(\alpha) | S]^2
\]
The process risk (cont’d)

Let $S'(x) = \text{survival function used to calculate premiums (can in particular coincide with } S'(x))$

Focus on two premium systems

- Upfront premium + annual premiums proportional to natural premiums; $\alpha = \text{quota pertaining to the upfront premium; then:}$

\[
\Pi(\alpha) = \alpha \mathbb{E}[Y | S']
\]

\[
\pi_k(\alpha) = (1 - \alpha) P_k^{[N]}; \quad k = 0, 1, \ldots
\]

Loss function:

\[
L_1(\alpha) = \sum_{k=0}^{K_y} \alpha P_k^{[N]} v^k - \Pi(\alpha)
\]

and then:

\[
L_1(\alpha) = \alpha \left( \sum_{k=0}^{K_y} P_k^{[N]} v^k - \mathbb{E}[Y | S'] \right)
\]
in particular we find:
\[ \text{Var}[L_1(\alpha)] \propto \alpha^2 \]

Note that

- the variance increases with \( \alpha \), i.e. with the amount of the upfront premium
- no upfront premium paid \((\alpha = 0) \Rightarrow \text{Var}[L_1(0)|S'] = 0 \Rightarrow \) balance between expected costs and premiums in each year and absence of mortality / longevity risk for the insurer

- Upfront premium + annual level premiums; then:
  \[ \Pi(\alpha) = \alpha \mathbb{E}[Y|S'] \]
  \[ \pi_k(\alpha) = \pi(\alpha); \quad k = 0, 1, \ldots \]

Loss function:
\[ L_2(\alpha) = \sum_{k=0}^{K_y} \left( P_k^{[N]} - \pi(\alpha) \right) \nu^k - \Pi(\alpha) \]
Denote with $\pi$ the annual premium corresponding to $\alpha = 0$; then

$$\pi(\alpha) = (1 - \alpha) \pi$$

and hence

$$L_2(\alpha) = \sum_{k=0}^{K_y} \left( P_k^{[N]} - \pi \right) v^k - \alpha \left( \mathbb{E}(Y|S') - \pi \sum_{k=0}^{K_y} v^k \right)$$

We find:

$$\mathbb{E}[L_2(\alpha)|S] = \sum_{t=1}^{+\infty} \left[ t-1 \mid 1 q_y \left( \sum_{k=0}^{t-1} \left( P_k^{[N]} - \pi \right) v^k - \alpha \left( \mathbb{E}[Y|S'] - \pi \sum_{k=0}^{t-1} v^k \right) \right) \right]$$
\[ \text{Var}[L_2(\alpha)|S] = \]
\[ = \sum_{t=1}^{+\infty} \left[ (t-1)q_y \left( \sum_{k=0}^{t-1} \left( P_k^{[N]} - \pi \right) v^k - \alpha \left( \mathbb{E}(Y|S') - \pi \sum_{k=0}^{t-1} v^k \right) \right) \right]^2 \]
\[ - \left( \sum_{t=1}^{+\infty} \left[ (t-1)q_y \left( \sum_{k=0}^{t-1} \left( P_k^{[N]} - \pi \right) v^k - \alpha \left( \mathbb{E}(Y|S') - \pi \sum_{k=0}^{t-1} v^k \right) \right) \right)^2 \]
Moments of the loss function at portfolio level

Loss functions at portfolio level, for a portfolio of (initially) \( N \) risks:

\[
\mathcal{L}_i(\alpha) = \sum_{j=1}^{N} L_i^{(j)}(\alpha); \quad i = 1, 2
\]

where \( L_i^{(j)}(\alpha) \) denotes the loss function of the insured \( j \)

In a portfolio of \( N \) homogeneous and (conditionally) independent risks:

\[
\mathbb{E}[\mathcal{L}_i(\alpha)|S] = N \mathbb{E}[L_i(\alpha)|S]
\]
\[
\text{Var}[\mathcal{L}_i(\alpha)|S] = N \text{Var}(L_i(\alpha)|S)
\]
The process risk (cont’d)

Portfolio valuations: riskiness and the portfolio size

Let \( Y \) = random present value of the benefits at portfolio level

Risk index (or coefficient of variation):

\[
r = \frac{\sigma[Y|S]}{\mathbb{E}[Y|S]}
\]

For a portfolio of homogeneous and independent risks:

\[
\mathbb{E}[Y|S] = N \mathbb{E}[Y|S]
\]
\[
\text{Var}[Y|S] = N \text{Var}[Y|S]
\]

Hence:

\[
r = \frac{1}{\sqrt{N}} \frac{\sigma[Y|S]}{\mathbb{E}[Y|S]}
\]

⇒ riskiness decreases as the portfolio size increases
Examples

Mortality assumptions:

\[ S^{[\text{min}]}(x), \ S^{[\text{med}]}(x), \ S^{[\text{max}]}(x) \]

(see the following table)

Assume:

- age at retirement \( y = 65 \)
- expected number of claims in the year of age \((x, x + 1)\)
  \[
  \phi_x = 0.1048 \times 0.272859 \times e^{0.029841 x}
  \]
- expected cost per claim at age \( x \), \( c_x = c = 1 \)
- rate of interest \( i = 0.03 \)
- mortality assumption for premium calculation \( S' = S^{[\text{med}]} \)
### The process risk (cont’d)

<table>
<thead>
<tr>
<th></th>
<th>$S<a href="x">\text{min}</a>$</th>
<th>$S<a href="x">\text{med}</a>$</th>
<th>$S<a href="x">\text{max}</a>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>83.50</td>
<td>85.20</td>
<td>87.00</td>
</tr>
<tr>
<td>$b$</td>
<td>8.00</td>
<td>9.15</td>
<td>10.45</td>
</tr>
<tr>
<td>$\xi$</td>
<td>82.118</td>
<td>84.129</td>
<td>86.167</td>
</tr>
<tr>
<td>$\mathbb{E}[T_0]$</td>
<td>78.636</td>
<td>80.742</td>
<td>82.920</td>
</tr>
<tr>
<td>$\mathbb{V}[T_0]$</td>
<td>136.120</td>
<td>111.560</td>
<td>91.577</td>
</tr>
</tbody>
</table>

**Three projected survival functions**

The following tables show:

- variance of the individual loss function conditional on $S[\text{med}]$
- riskiness for portfolio size $N = 100$ and $N = 10000$
The process risk (cont’d)

| $\alpha$ | $\text{Var}[L_1(\alpha)|S]$ | $\text{Var}[L_2(\alpha)|S]$ |
|----------|-----------------------------|-----------------------------|
| 0.0      | 0.00000                     | 0.14071                     |
| 0.1      | 0.02757                     | 0.23755                     |
| 0.2      | 0.11029                     | 0.37103                     |
| 0.3      | 0.24816                     | 0.54113                     |
| 0.4      | 0.44118                     | 0.74785                     |
| 0.5      | 0.68935                     | 0.99121                     |
| 0.6      | 0.99266                     | 1.27119                     |
| 0.7      | 1.35112                     | 1.58780                     |
| 0.8      | 1.76473                     | 1.94103                     |
| 0.9      | 2.23348                     | 2.33089                     |
| 1.0      | 2.75738                     | 2.75738                     |

Variance of the loss function
The process risk (cont’d)

<table>
<thead>
<tr>
<th></th>
<th>$N = 100$</th>
<th></th>
<th>$N = 10000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mathbb{E}[Y</td>
<td>S]$</td>
<td>$\text{Var}[Y</td>
</tr>
<tr>
<td>$S^{[\text{min}]}(x)$</td>
<td>337.733</td>
<td>295.406</td>
<td>0.0509</td>
</tr>
<tr>
<td>$S^{[\text{med}]}(x)$</td>
<td>357.715</td>
<td>275.738</td>
<td>0.0464</td>
</tr>
<tr>
<td>$S^{[\text{max}]}(x)$</td>
<td>384.815</td>
<td>256.930</td>
<td>0.0417</td>
</tr>
</tbody>
</table>

Riskiness for two portfolio sizes
6.6 THE UNCERTAINTY RISK

**Portfolio valuations: moments of the loss function**

Assign the probabilities

\[ \rho_{[\text{min}]}, \rho_{[\text{med}]}, \rho_{[\text{max}]} \]

to the survival functions \( S_{[\text{min}]}(x), S_{[\text{med}]}(x), S_{[\text{max}]}(x) \) respectively.

Unconditional expected value and variance of loss function

\[
\mathbb{E}[\mathcal{L}_i(\alpha)] = \mathbb{E}_\rho[\mathbb{E}[\mathcal{L}_i(\alpha)|S]) = N \mathbb{E}_\rho[\mathbb{E}[L_i(\alpha)|S]] = N \mathbb{E}[L_i(\alpha)]; \quad i = 1, 2
\]

\[
\text{Var}[\mathcal{L}_i(\alpha)] = \mathbb{E}_\rho[\text{Var}[\mathcal{L}_i(\alpha)|S]] + \text{Var}_\rho[\mathbb{E}[\mathcal{L}_i(\alpha)|S]]
\]

\[
= N \mathbb{E}_\rho[\text{Var}[L_i(\alpha)|S]] + N^2 \text{Var}_\rho[\mathbb{E}[L_i(\alpha)|S]]; \quad i = 1, 2
\]

- random fluctuations
- systematic deviations
The uncertainty risk (cont’d)

If $N = \bar{N}$, with

$$\bar{N} = {E_\rho[\text{Var}[L_i(\alpha)\mid S]] \over \text{Var}_\rho[\mathbb{E}[L_i(\alpha)\mid S]]}, \quad i = 1, 2$$

the two terms of the variance are equal

**Portfolio valuations: riskiness and the portfolio size**

Risk index:

$$r = {\sigma[Y] \over \mathbb{E}[Y]} = \left( {1 \over \bar{N}} {E_\rho[\text{Var}[Y\mid S]] \over \mathbb{E}^2[Y]} + {\text{Var}_\rho[\mathbb{E}[Y\mid S]] \over \mathbb{E}^2[Y]} \right)^{1/2}$$

- diversifiable
- non-diversifiable
The uncertainty risk (cont’d)

**Examples**

Assume

\[ \rho^{[\text{min}]} = 0.2, \quad \rho^{[\text{med}]} = 0.6, \quad \rho^{[\text{max}]} = 0.2 \]

Other data as in the previous example

The following tables show:

- Expected value, variance and relevant components, in the case of premiums proportional to annual expected costs
- Expected value, variance and relevant components, in the case of level premiums
- Expected value, variance and risk index as functions of the portfolio size
The uncertainty risk (cont’d)

| α  | $\mathbb{E}[L_1(\alpha)]$ | $\text{Var}[L_1(\alpha)]$ | $\mathbb{E}_\rho[\text{Var}[L_1(\alpha)|S]]$ | $\text{Var}_\rho[\mathbb{E}[L_1(\alpha)|S]]$ | $\bar{N}$ |
|----|-----------------------------|-----------------------------|-----------------------------------------------|-----------------------------------------------|--------|
| 0.0| 0.000                       | 0.000                       | 0.000                                         | 0.000                                         | –      |
| 0.1| 14.237                      | 22 747.223                  | 275.910                                       | 22 471.313                                    | 122.783|
| 0.2| 28.473                      | 90 988.893                  | 1 103.641                                     | 89 885.252                                    | 122.783|
| 0.3| 42.710                      | 204 725.009                 | 2 483.192                                     | 202 241.817                                   | 122.783|
| 0.4| 56.947                      | 363 955.571                 | 4 414.563                                     | 359 541.008                                   | 122.783|
| 0.5| 71.183                      | 568 680.580                 | 6 897.755                                     | 561 782.826                                   | 122.783|
| 0.6| 85.420                      | 818 900.036                 | 9 932.767                                     | 808 967.269                                   | 122.783|
| 0.7| 99.657                      | 1 114 613.938               | 13 519.599                                    | 1 101 094.338                                 | 122.783|
| 0.8| 113.893                     | 1 455 822.286               | 17 658.252                                    | 1 438 164.034                                 | 122.783|
| 0.9| 128.130                     | 1 842 525.081               | 22 348.725                                    | 1 820 176.355                                 | 122.783|
| 1.0| 142.367                     | 2 274 722.322               | 27 591.019                                    | 2 247 131.303                                 | 122.783|

Expected value, variance and relevant components
(premiums proportional to annual expected costs)
### The uncertainty risk (cont’d)

| \( \alpha \) | \( \mathbb{E} \{ \mathcal{L}_2(\alpha) \} \) | \( \text{Var} \{ \mathcal{L}_2(\alpha) \} \) | \( \mathbb{E}_\rho \left[ \text{Var} \{ \mathcal{L}_2(\alpha) | S \} \right] \) | \( \text{Var}_\rho \left[ \mathbb{E} \{ \mathcal{L}_2(\alpha) | S \} \right] \) | \( \bar{N} \) |
|---|---|---|---|---|---|
| 0.0 | 42.365 | 54,388.611 | 1,430.150 | 52,958.461 | 270.051 |
| 0.1 | 52.365 | 129,488.780 | 2,407.495 | 127,081.285 | 189.445 |
| 0.2 | 62.365 | 237,240.772 | 3,749.004 | 233,491.767 | 160.563 |
| 0.3 | 72.365 | 377,644.586 | 5,454.679 | 372,189.907 | 146.556 |
| 0.4 | 82.366 | 550,700.223 | 7,524.518 | 543,175.704 | 138.528 |
| 0.5 | 92.366 | 756,407.682 | 9,958.523 | 746,449.160 | 133.412 |
| 0.6 | 102.366 | 994,766.965 | 12,756.692 | 982,010.273 | 129.904 |
| 0.7 | 112.366 | 1,265,778.070 | 15,919.026 | 1,249,859.044 | 127.367 |
| 0.8 | 122.366 | 1,569,440.998 | 19,445.525 | 1,549,995.472 | 125.455 |
| 0.9 | 132.366 | 1,905,755.748 | 23,336.190 | 1,882,419.559 | 123.969 |
| 1.0 | 142.367 | 2,274,722.322 | 27,591.019 | 2,247,131.303 | 122.783 |

---

*Expected value, variance and relevant components (level premiums)*
The uncertainty risk (cont’d)

| $N$  | $\mathbb{E}[\mathcal{Y}]$ | $\text{Var}[\mathcal{Y}]$ | $\mathbb{E}_{\rho}\left[\text{Var}[\mathcal{Y}|\mathcal{S}]\right]$ | $\text{Var}_{\rho}\left[\mathbb{E}[\mathcal{Y}|\mathcal{S}]\right]$ | $r = \frac{\sigma[\mathcal{Y}]}{\mathbb{E}[\mathcal{Y}]}$ |
|------|-----------------|-----------------|------------------|------------------|-----------------|
| 100  | 359.139         | 500.623         | 275.910          | 224.713          | 0.062           |
| 200  | 718.278         | 1450.673        | 551.820          | 898.853          | 0.053           |
| 1000 | 3591.388        | 25 230.415      | 2759.102         | 22471.313        | 0.044           |
| 10000| 35 913.882      | 2 274 722.322   | 27 591.019       | 22 471 313.03    | 0.042           |
| $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 0.042 |

*Expected value, variance and risk index as functions of the portfolio size*
6.7 PREMIUM LOADINGS

*Premium loading and loss function*

From previous Section: arrangements where annual premiums are proportional to annual expected costs are less risky than systems with level annual premiums

However, level premiums may be preferred

In order to design appealing premium systems, but aiming at limiting risk ⇒ level premiums charged with an appropriate safety loading

Let \( \pi(\alpha; \lambda) \) = charged premium

Assume a proportional loading:

\[
\pi(\alpha; \lambda) = (1 + \lambda) \pi(\alpha)
\]

For this premium arrangement:

- \( L_3(\alpha) \) = individual loss function
- \( L_3(\alpha) \) = portfolio loss function


**Premium loadings (cont’d)**

\[
L_3(\alpha) = \sum_{k=0}^{K_y} \left( P_k^{[N]} - \pi(\alpha; \lambda) \right) u^k - \Pi(\alpha)
\]

\[
\mathcal{L}_3(\alpha) = \sum_{j=1}^{N} L_3^{(j)}(\alpha)
\]

Reasonable aims:

\[
\text{Var}(L_3(\alpha)) = \text{Var}(L_1(\alpha)) \quad (\ast)
\]
\[
\text{Var}(\mathcal{L}_3(\alpha)) = \text{Var}(\mathcal{L}_1(\alpha)) \quad (\ast\ast)
\]

where \(L_1(\alpha), \mathcal{L}_1(\alpha)\) relate to annual premiums proportional to natural premiums.

Equations (\ast), (\ast\ast) \Rightarrow charge premiums so that the variance of the loss function is the lowest within the probabilistic structure adopted, for a given upfront premium.
**Process risk**

Given the link between the variance of the loss function at individual and portfolio level, loadings resulting from requirements (*) and (**) coincide ⇒ focus on the individual case only

It can be proved that, because of the expression of $\text{Var}(L_3(\alpha)|S)$, Eq. (*) has the structure

$$A \lambda^2 + B \lambda + C = 0$$

In the following table:

- $S^{[\text{med}]}$ has been adopted
- when no real solution for equation (*) exists, $\lambda$ has been set equal to the minimum point $\lambda^*$ of the function

$$f(\lambda) = A \lambda^2 + B \lambda + C$$
- when the equation is possible, the lower solution has been chosen
### Premium loadings (cont’d)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\lambda$</th>
<th>$\lambda^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.2144</td>
<td>0.2144</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3493</td>
<td>0.3493</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3038</td>
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<td>0.3</td>
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<td>0.7349</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2602</td>
<td>1.0240</td>
</tr>
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<td>0.2531</td>
<td>1.4288</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2486</td>
<td>2.0360</td>
</tr>
<tr>
<td>0.7</td>
<td>0.2454</td>
<td>3.0480</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2431</td>
<td>5.0721</td>
</tr>
<tr>
<td>0.9</td>
<td>0.2413</td>
<td>11.1441</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0000</td>
<td>–</td>
</tr>
</tbody>
</table>

*Solutions of the loading equation (process risk only)*
Uncertainty risk

Eq. (***) must be used
Loading parameter $\lambda$ depends on the size $N$ of the portfolio
It can be proved that, because of the expression of $\text{Var}(\mathcal{L}_3(\alpha))$, Eq. (***) has the structure

$$A(N) \lambda^2 + B(N) \lambda + C(N) = 0$$

Coefficients $A(N)$, $B(N)$, $C(N)$ are second order polynomials with respect to $N$

In the following table:

- when no real solution for equation (***) exists, $\lambda$ has been set equal to the minimum point $\lambda^*$ of the function

$$g(\lambda, N) = A(N) \lambda^2 + B(N) \lambda + C(N)$$

- when the equation is possible, the lower solution has been chosen
As $N$ increases $\Rightarrow$ random fluctuation component tends to vanish $\Rightarrow$ the required premium loading decreases and has a positive limit

<table>
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<tr>
<th>$\alpha$</th>
<th>$N = 1$</th>
<th>$N = 100$</th>
<th>$N = 1000$</th>
<th>$N = 10000$</th>
<th>$N = 100000$</th>
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<tr>
<td></td>
<td>$\lambda$</td>
<td>$\lambda^*$</td>
<td>$\lambda$</td>
<td>$\lambda^*$</td>
<td>$\lambda$</td>
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<td>0.2157</td>
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<td>0.1989</td>
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<td>0.2321</td>
<td>0.7127</td>
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<td>0.2216</td>
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<td>$-$</td>
<td>0.0000</td>
<td>$-$</td>
<td>0.0000</td>
</tr>
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</table>

**Solutions of the loading equation (process risk & uncertainty risk)**