Actuaries Institute

Pricing tree structures with shifting parameters

Greg Taylor
UNSW Business School

Insights, 23 November 2017

Actuaries Institute

Acknowledgements and disclaimers

- Principal financial support for this research provided by a grant from the Australian Actuaries Institute
- Also supported under Australian Research Council's Linkage Projects funding scheme (project number LP130100723)
- The views expressed herein are those of the author and are not necessarily those of either supporting body

Overview

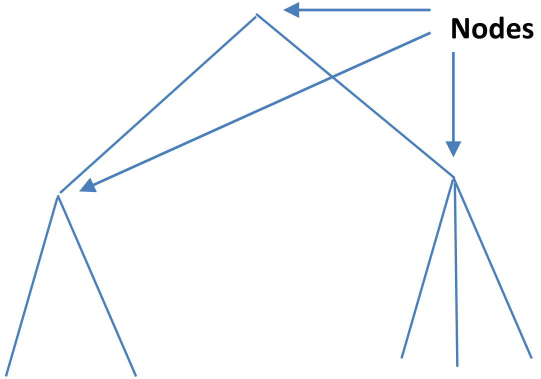
- Motivation
- Formal framework
- Hierarchical models
  - Static
  - Evolutionary
- Kalman filter
  - In general
  - Application to hierarchical model
- Numerical example
- Conclusion

Overview

- **Motivation**
- Formal framework
- Hierarchical models
  - Static
  - Evolutionary
- Kalman filter
  - In general
  - Application to hierarchical model
- Numerical example
- Conclusion

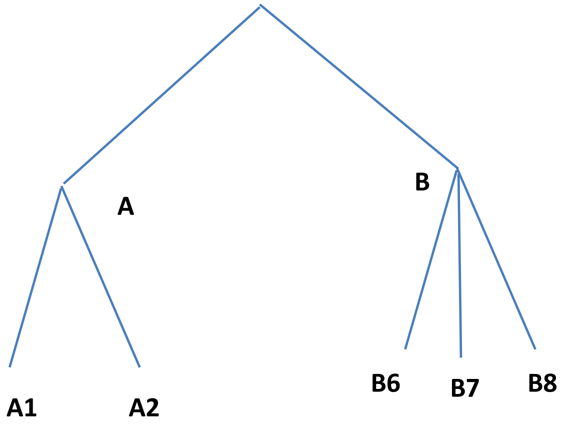
Motivation: hierarchies (1)

- What is a hierarchy?



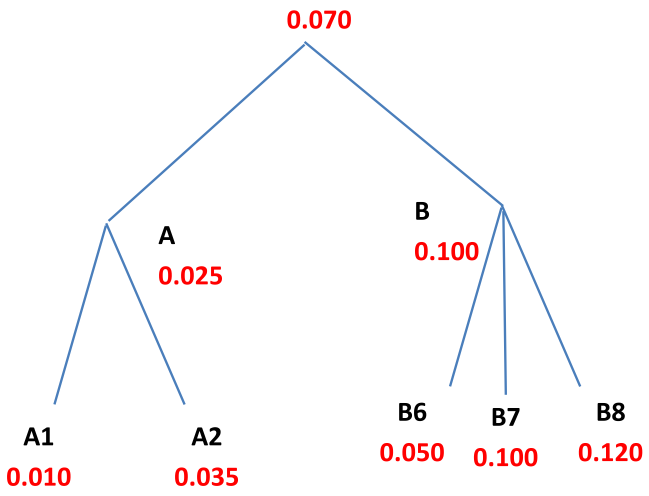
Motivation: hierarchies (2)

- Example: the ANZSIC occupational codes
  - A = Agriculture, Forestry and Fishing
    - A1 = Agriculture
    - A2 = Aquaculture
  - B = Mining
    - B6 = Coal mining
    - B7 = Oil and gas extraction
    - B8 = Metal ore mining



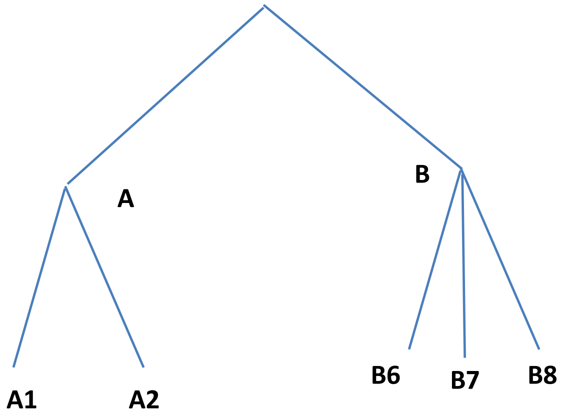
Motivation: hierarchies (3)

- There may be an observation at each node
  - e.g. claim frequency
- One may wish to estimate the claim frequency parameters underlying these observations
- One can use **hierarchical credibility**
  - Taylor (1979)
  - Sundt (1979,1980)



Motivation: static versus evolutionary hierarchies

- Suppose that claim frequencies at the node **evolve** over time
- Now a parameter estimate is required at each node **at each point of time**
- How?



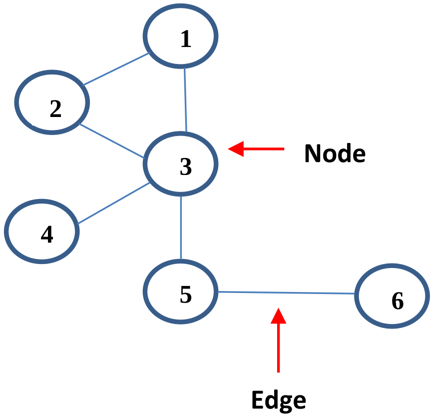
Overview

- Motivation
- **Formal framework**
- Hierarchical models
  - Static
  - Evolutionary
- Kalman filter
  - In general
  - Application to hierarchical model
- Numerical example
- Conclusion

Graph theory (1)

- A **graph** $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$ consisting of a set $V(\mathcal{H})$ of **vertices**, or **nodes**, and a set $E(\mathcal{H})$, disjoint from $V(\mathcal{H})$, of **edges**, together with an **incidence matrix** $\Psi(\mathcal{H})$ that associates with each edge of $\mathcal{H}$ an unordered pair of (not necessarily distinct) nodes of $\mathcal{H}$.

Example



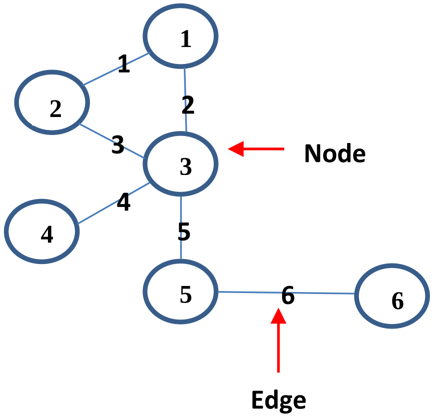
Graph theory (2)

- Incidence matrix

Node	Edge					
	1	2	3	4	5	6
1	1	1	0	0	0	0
2	1	0	1	0	0	0
3	0	1	1	1	1	0
4	0	0	0	1	0	0
5	0	0	0	0	1	1
6	0	0	0	0	0	1

  - Edge 1 is incident with nodes 1 and 2

Example

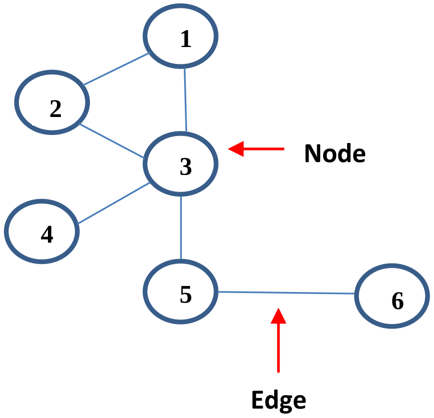


Graph theory (3)

- **Adjacency matrix**
  - Indicates, for each node, which nodes are adjacent, i.e. related by a single edge

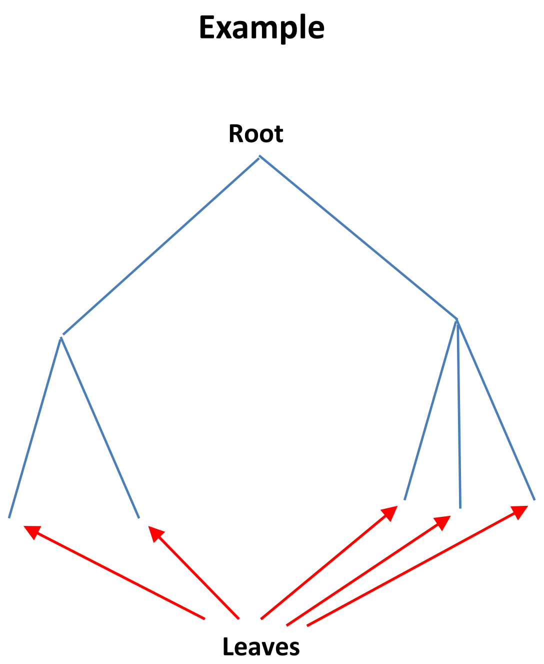
Node	Node					
	1	**2**	**3**	**4**	**5**	**6**
1	0	1	1	0	0	0
2	1	0	1	0	0	0
3	1	1	0	1	1	0
4	0	0	1	0	0	0
5	0	0	1	0	0	1
6	0	0	0	0	1	0

Example



Graph theory: tree

- A **tree** is special case of a graph in which any two nodes are connected by exactly one path
- A **hierarchy** is the same as a tree



Adjacency matrix for tree

Entire graph

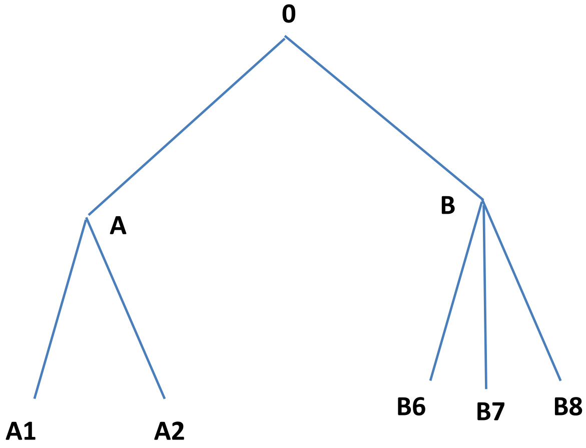
	0	A	B	A1	A2	B6	B7	B8
0	0	1	1	0	0	0	0	0
A	0	0	0	1	1	0	0	0
B	0	0	0	0	0	1	1	1
A1	0	0	0	0	0	0	0	0
A2	0	0	0	0	0	0	0	0
B6	0	0	0	0	0	0	0	0
B7	0	0	0	0	0	0	0	0
B8	0	0	0	0	0	0	0	0

- Graph is now **directed**
- Denote $\Gamma(\mathcal{H})$

Sub-graph relating two levels

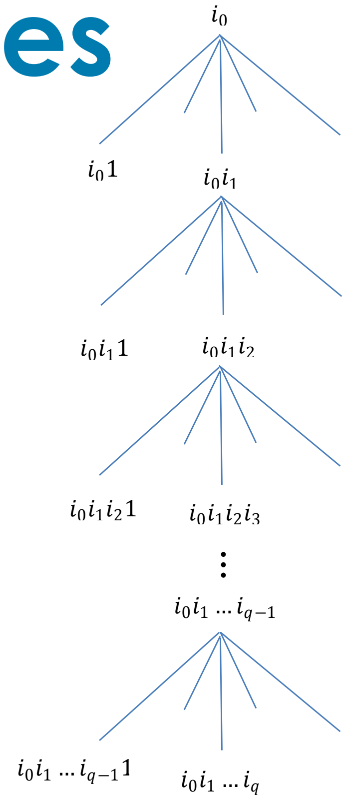
	A1	A2	B6	B7	B8
A	1	1	0	0	0
B	0	0	1	1	1

- Denote $\Gamma(\mathcal{H}_2)$ where $\mathcal{H}_2$ is the sub-hierarchy consisting of just levels 1 and 2



Larger hierarchies (trees): labelling of nodes

- This hierarchy consists of a root (labelled $i_0 = 1$) and q **levels**
  - This a ***q*-hierarchy**
- Nodes at level m are labelled $i_0 i_1 \ldots i_{m-1}\, i_m, i_n = 1,2, etc., n = 1,2, \ldots, m$

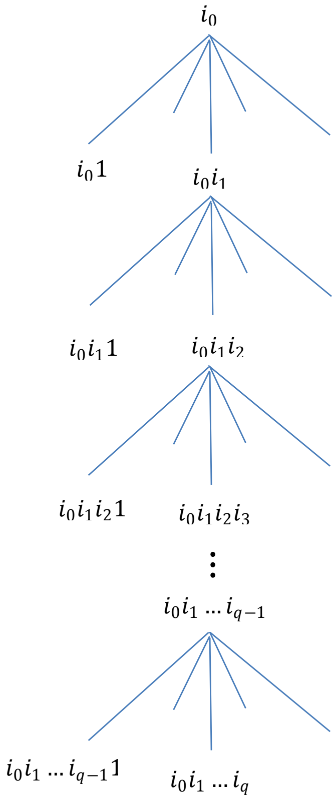


Overview

- Motivation
- Formal framework
- **Hierarchical models**
  - **Static**
  - Evolutionary
- Kalman filter
  - In general
  - Application to hierarchical model
- Numerical example
- Conclusion

Hierarchical model: notation

- Associate a parameter vector with each node
  - Parameter vector at node $i_0 i_1 \ldots i_{m-1}\, i_m$ denoted $\beta_{i_0 i_1 \ldots i_m}$
  - $\beta_{(m)} = \begin{bmatrix} \beta_{i_0 i_1 \ldots i_{m-1} 1} \\ \beta_{i_0 i_1 \ldots i_{m-1} 2} \\ \vdots \end{bmatrix}$ denotes vector of all parameters at level m
  - $\beta = \begin{bmatrix} \beta_{(0)} \\ \beta_{(1)} \\ \vdots \\ \beta_{(q)} \end{bmatrix}$ denotes vector of all parameters at all levels



Static hierarchical model: formal statement

- Simplified version of Sundt's (1980) model
  - A parameter vector $\beta_{i_0 i_1 \dots i_m}$ is associated with node $i_0 i_1 \dots i_m$ of the hierarchy
  - The parameter β_{i_0} at the root of the tree is fixed
  - For $m = 0,1, \dots, q-1$, the parameter vector $\beta_{i_0 i_1 \dots i_m i_{m+1}}$ is a random drawing from some distribution determined by $\beta_{i_0 i_1 \dots i_m}$
  - At each of the hierarchy's leaves $i_0 i_1 \dots i_q$ there exists a sample of observations $y_{i_0 i_1 \dots i_q j}, j = 1,2, \dots$ drawn from some distribution determined by $\beta_{i_0 i_1 \dots i_q}$

Static hierarchical model: formal statement (cont'd)

- The random parameters and observations are subject to the following dependency structure:
  - $\beta_{(m)} = W_{(m-1)}\beta_{(m-1)} + \zeta_{(m)}, m = 1, \ldots q;$
  - $y = X\beta_{(q)} + \varepsilon;$

where X is a design matrix, $W_{(m-1)}$ is some matrix (called a **transmission matrix**) compatible with the dimensions of $\beta_{(m-1)}$ and $\beta_{(m)}$, and $\zeta_{(m)}, \varepsilon$ are random vectors, with ε independent of the $\zeta_{(m)}$, and

- $E[\zeta] = 0, E[\varepsilon] = 0;$
- $Var[\zeta] = \Lambda, Var[\varepsilon] = H;$

where ζ is the vector obtained by stacking the $\zeta_{(m)}$

Overview

- Motivation
- Formal framework
- Hierarchical models
  - Static
  - **Evolutionary**
- Kalman filter
  - In general
  - Application to hierarchical model
- Numerical example
- Conclusion

Evolutionary hierarchical model: formal statement

- Consider a q-hierarchy $\mathcal{H}$, supplemented by parameters and observations that satisfy the following conditions. At each time $t = 0,1, \ldots$:
  - A parameter vector $\beta^t_{i_0 i_1 \ldots i_m}$ is associated with node $i_0 i_1 \ldots i_m$ of the hierarchy.
  - For $m = 0,1, \ldots, q-1$, the parameter vector $\beta^t_{i_0 i_1 \ldots i_m i_{m+1}}$ is a random drawing from some distribution determined by $\beta^t_{i_0 i_1 \ldots i_m}$.
  - At each of the hierarchy's terminal nodes $i_0 i_1 \ldots i_q$ there exists a sample of observations $y^t_{i_0 i_1 \ldots i_q j}, j = 1,2, \ldots$ drawn from some distribution determined by $\beta^t_{i_0 i_1 \ldots i_q}$.

Evolutionary hierarchical model: formal statement (cont'd)

- The observations are subject to the following dependency on parameters:

$$y^t = X^t \beta^t_{(q)} + \varepsilon^t,$$

where X^t is a design matrix, ε^t is a random vector, and

$$E[\varepsilon^t] = 0, Var[\varepsilon^t] = H^t.$$

- The parameter vector β^t evolves over time as follows. Define $\gamma^t_{(0)} = \beta^t_{(0)}$ and $\gamma^t_{(m)} = \beta^t_{(m)} - W_{(m-1)}\beta^t_{(m-1)}, m = 1, \dots, q$

Evolutionary hierarchical model: formal statement (cont'd)

Assume that:

- The parameter vector β^0 at $t = 0$ is random with known $E[\beta^0], Var[\beta^0]$
- The parameters $\gamma_{(m)}^t$ evolve according to:

$$\gamma_{(m)}^t = \gamma_{(m)}^{t-1} + \zeta_{(m)}^t, m = 0, \ldots q; t = 1,2, \ldots,$$

where $\zeta_{(m)}^t$ is a random vector, and

$$E[\zeta^t] = 0, Var[\zeta^t] = \Lambda^t,$$

and where ζ^t is the vector obtained by stacking the $\zeta_{(m)}^t$ and

all $\zeta^t, \varepsilon^t, t = 0,1,2, \ldots$ are mutually independent.

Alternative form of model

Original model

$$y^t = X^t \beta^t_{(q)} + \varepsilon^t$$

Alternative statement

$$y^t = U^t \gamma^t + \varepsilon^t$$

where

$$U^t = [X^t W_{(0:q)} \quad X^t W_{(1:q)} \cdots \quad X^t W_{(q:q)}]$$

and

$$W_{(m:m+p)} = W_{(m+p-1)} W_{(m+p-2)} \ldots W_{(m)}$$

Overview

- Motivation
- Formal framework
- Hierarchical models
  - Static
  - Evolutionary
- **Kalman filter**
  - **In general**
  - Application to hierarchical model
- Numerical example
- Conclusion

Kalman filter (slightly simplified): model formulation

- At each $t = 0,1,\ldots$
  - **Observation** (or **measurement**) **equation**
    $$y^t = U^t\gamma^t + \varepsilon^t$$
  - **System** (or **transition**) **equation**
    $$\gamma^t = \gamma^{t-1} + \zeta^t, t = 1,2,\ldots$$
- The parameter vector γ^0 at $t = 0$ is random with known $E[\gamma^0], Var[\gamma^0]$
- $\varepsilon^t \sim N(0, H^t), \zeta^t \sim N(0, \Lambda^t)$, with all $\zeta^t, \varepsilon^t, t = 0,1,2,\ldots$ are mutually independent

Kalman filter: parameter estimation algorithm

- Let $\gamma^{t|s}$ and $P^{t|s}$ denote the estimators of $\gamma^t, Var[\gamma^t - \gamma^{t|s}]$ given data $\{y^0, y^1, \dots, y^s\}$. The filter comprises the following procedure for each $t = 1,2, \dots$:
  1) Commence with estimate $\gamma^{t|t-1}$ and covariance matrix $P^{t|t-1}$ of the same dimension
  2) Calculate $F^t = U^t P^{t|t-1} (U^t)^T + H^t$
  3) Calculate $K^t = P^{t|t-1} (U^t)^T (F^t)^{-1}$, called the **Kalman gain matrix**
  4) Update the matrix $P^{t|t-1}$ as follows:
  $$P^{t+1|t} = P^{t|t-1} - P^{t|t-1} (U^t)^T (F^t)^{-1} U^t P^{t|t-1} + \Lambda^t$$
  5) Update the estimate $\gamma^{t|t-1}$ as follows: $\gamma^{t|t} = \gamma^{t|t-1} + K^t \left(y^t - U^t \gamma^{t|t-1} \right)$
  6) Further update $\gamma^{t|t}$ as follows: $\gamma^{t+1|t} = \gamma^{t|t}$

Credibility type of estimator

Overview

- Motivation
- Formal framework
- Hierarchical models
  - Static
  - Evolutionary
- Kalman filter
  - In general
  - **Application to hierarchical model**
- Numerical example
- Conclusion

Comparison of evolutionary hierarchical and Kalman filter models

Evolutionary hierarchical model

$$y^t = U^t \gamma^t + \varepsilon^t$$
$$\gamma^t_{(m)} = \gamma^{t-1}_{(m)} + \zeta^t_{(m)}$$

Kalman filter model

$$y^t = U^t \gamma^t + \varepsilon^t$$
$$\gamma^t = \gamma^{t-1} + \zeta^t$$

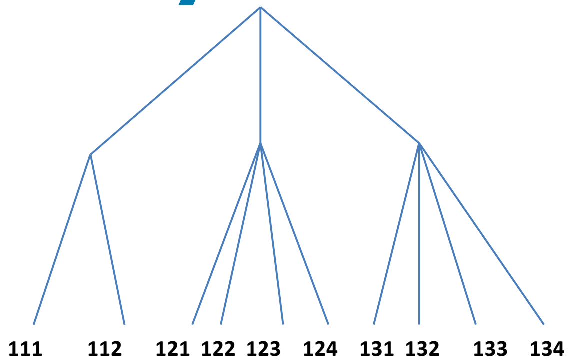
- So Kalman filter can be applied to estimate γ^t for each t
  - i.e. β^t for each t

Overview

- Motivation
- Formal framework
- Hierarchical models
  - Static
  - Evolutionary
- Kalman filter
  - In general
  - Application to hierarchical model
- **Numerical example**
- Conclusion

Numerical example: definition of hierarchy

- **Level 0:** Single node {1}
- **Level 1:** Nodes {11,12,13}
- **Level 2:** Nodes {111,112,121,122, 123,124,131,132, 133,134}



- $y^t_{i_0 i_1 i_2}$, **claim frequencies per unit exposure** $(E^t_{i_0 i_1 i_2})$ observed at the leaves

Numerical example: model structure

- Dependency of observations on parameters

$$y^t = X^t \beta^t_{(2)} + \varepsilon^t \text{ with } X^t = \mathrm{I}$$

$$\text{i.e. } E\left[y^t_{i_0 i_1 i_2}\right] = \beta^t_{i_0 i_1 i_2}$$

- Dependencies between levels of hierarchy

$$W_{(m)} = [\Gamma(\mathcal{H}_m)]^T, \text{ i.e. } E\left[\beta^t_{i_0 \ldots i_{m+1}}\right] = \beta^t_{i_0 \ldots i_m}$$

- Variances and covariances

$$Var\left[y^t_{i_0 i_1 i_2}\right] = \beta^t_{i_0 i_1 i_2} / E^t_{i_0 i_1 i_2}$$

  - Covariance matrices Λ^t, H^t diagonal

Numerical example: true parameters

Node	Initial values (β^0)		Parameter variance (Λ^t)
	Mean	Variance	
1	0.070	0.00005	0.00005
11	0.025	0.00003	0.00001
12	0.100	0.00030	0.00005
13	0.150	0.00070	0.00015
111	0.010	0.00002	0.00001
112	0.035	0.00015	0.00002
121	0.050	0.00040	0.00004
122	0.080	0.00090	0.00010
123	0.100	0.00150	0.00015
124	0.120	0.00250	0.00025
131	0.135	0.00300	0.00030
132	0.155	0.00400	0.00040
133	0.180	0.00500	0.00050
134	0.200	0.00650	0.00070

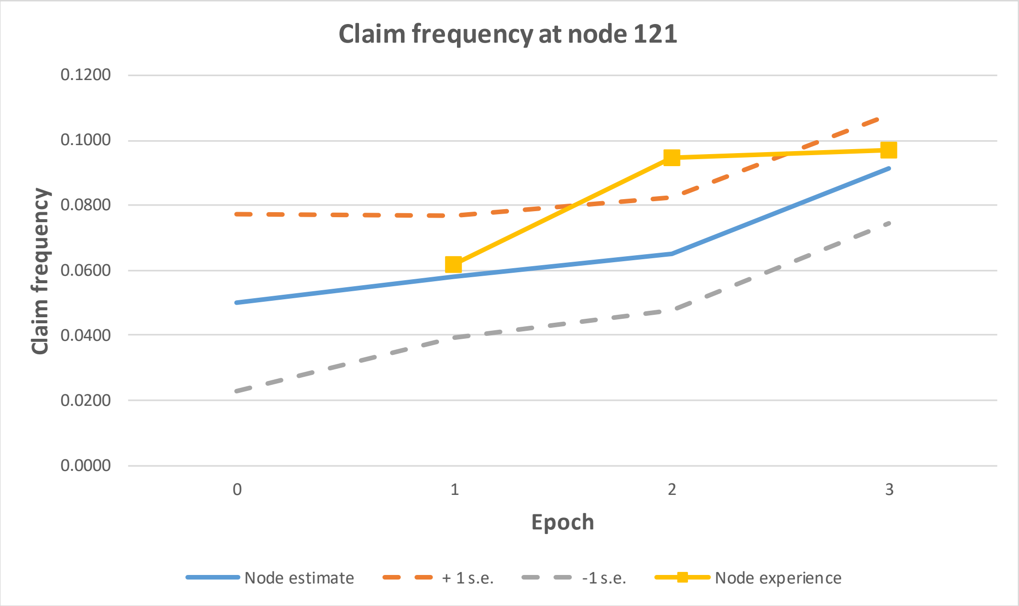
Numerical example: simulated data

Node	Exposure ($E^{t}_{i_0 i_1 \dots i_q}$)	Observed claim frequency at t =		
		1	2	3
111	40	0.007	0.013	0.007
112	35	0.030	0.038	0.043
121	300	0.062	0.094	0.097
122	100	0.081	0.088	0.079
123	500	0.120	0.064	0.136
124	100	0.093	0.053	0.081
131	301	0.150	0.143	0.132
132	50	0.172	0.136	0.093
133	25	0.111	0.188	0.094
134	20	0.248	0.171	0.195

- All nodes trendless except:
  - **Node 121:** upward trend of 0.015 per period added
  - **Node 124:** flat reduction of 0.040 made in each period
  - **Node 132:** downward trend of 0.020 per period added

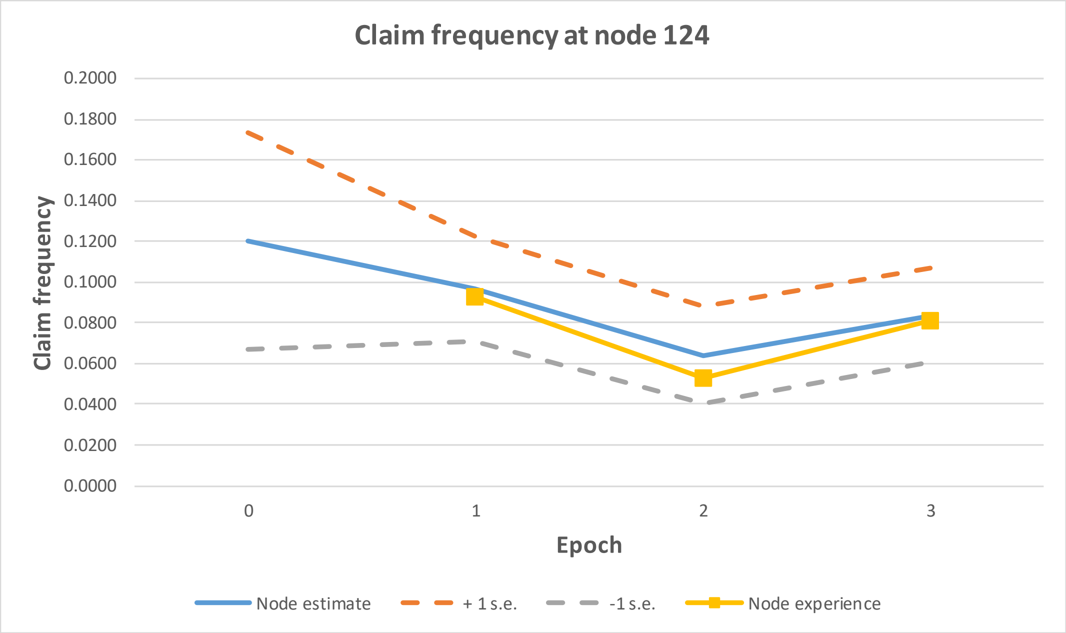
Numerical example: results (1)

- Node 121: increasing trend in true frequency



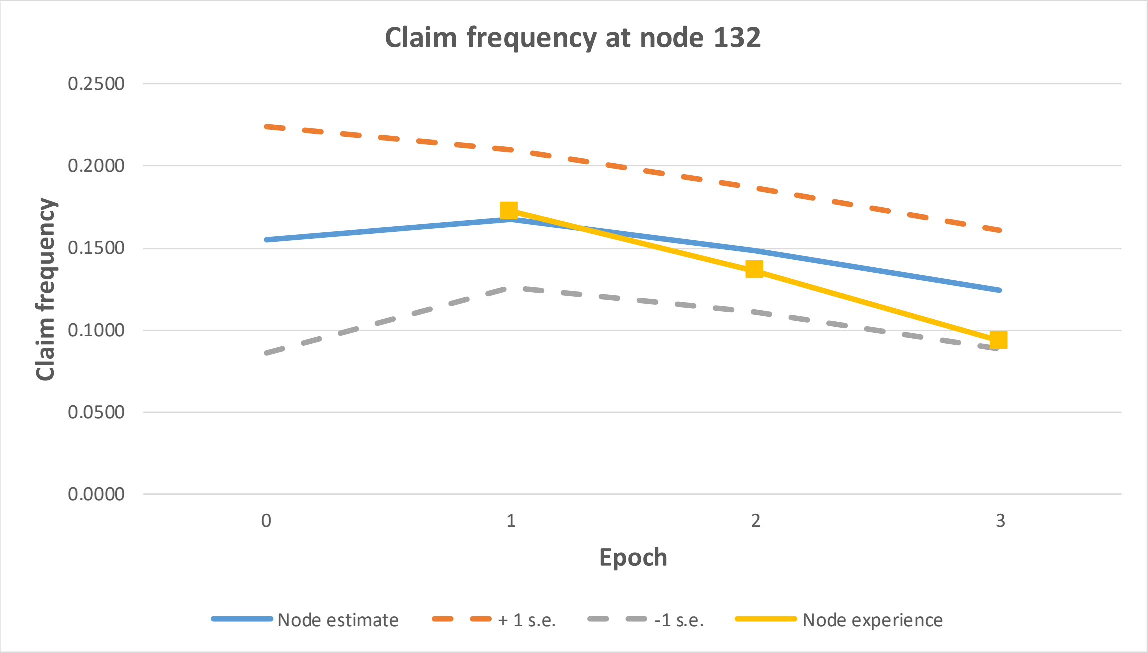
Numerical example: results (2)

- Node 124: one-off downward shift in true frequency at $t = 1$



Numerical example: results (3)

- Node 132: decreasing trend in true frequency



Overview

- Motivation
- Formal framework
- Hierarchical models
  - Static
  - Evolutionary
- Kalman filter
  - In general
  - Application to hierarchical model
- Numerical example
- **Conclusion**

Conclusion

- Evolutionary hierarchical model has been formulated
- Parameter estimates obtained by application of the Kalman filter
  - Filter updates estimates from one epoch to the next as further data are observed
- Application of the Kalman filter conceptually straightforward, but
  - Tree structure of the model parameters can be extensive
  - Some effort is required to retain organization of the updating algorithm
  - Achieved by suitable manipulation of the adjacency matrix associated with the tree
  - Adjacency matrix then recruited to play its role in Kalman filter matrix calculations

References

- Sundt B (1979). A hierarchical credibility regression model. **Scandinavian Actuarial Journal**, 107-114.
- Sundt B (1980). A multi-level hierarchical credibility regression model. **Scandinavian Actuarial Journal**, 25-32.
- Taylor G C (1979). Credibility analysis of a general hierarchical model, **Scandinavian Actuarial Journal**, 1979(1), 1-12.

Questions?