Pricing tree structures with shifting parameters

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• The views expressed herein are those of the author and are not necessarily those of either supporting body
Overview

• Motivation
• Formal framework
• Hierarchical models
  – Static
  – Evolutionary
• Kalman filter
  – In general
  – Application to hierarchical model
• Numerical example
• Conclusion
Overview

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Motivation: hierarchies (1)

• What is a hierarchy?
Motivation: hierarchies (2)

- Example: the ANZSIC occupational codes
  - A = Agriculture, Forestry and Fishing
    - A1 = Agriculture
    - A2 = Aquaculture
  - B = Mining
    - B6 = Coal mining
    - B7 = Oil and gas extraction
    - B8 = Metal ore mining
Motivation: hierarchies (3)

• There may be an observation at each node
  – e.g. claim frequency
• One may wish to estimate the claim frequency parameters underlying these observations
• One can use hierarchical credibility
  – Taylor (1979)
  – Sundt (1979,1980)
Motivation: static versus evolutionary hierarchies

• Suppose that claim frequencies at the node evolve over time
• Now a parameter estimate is required at each node at each point of time
• How?
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Graph theory (1)

- A graph $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$ consisting of a set $V(\mathcal{H})$ of vertices, or nodes, and a set $E(\mathcal{H})$, disjoint from $V(\mathcal{H})$, of edges, together with an incidence matrix $\Psi(\mathcal{H})$ that associates with each edge of $\mathcal{H}$ an unordered pair of (not necessarily distinct) nodes of $\mathcal{H}$.
Graph theory (2)

• Incidence matrix

<table>
<thead>
<tr>
<th>Node</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
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<td>0</td>
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</tr>
</tbody>
</table>

– Edge 1 is incident with nodes 1 and 2
Graph theory (3)

- **Adjacency matrix**
  - Indicates, for each node, which nodes are adjacent, i.e. related by a single edge

<table>
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<tr>
<th>Node</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>1</td>
<td>1</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Graph theory: tree

- A **tree** is a special case of a graph in which any two nodes are connected by exactly one path.
- A **hierarchy** is the same as a tree.

![Example](image-url)
Adjacency matrix for tree

### Entire graph

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>A</th>
<th>B</th>
<th>A1</th>
<th>A2</th>
<th>B6</th>
<th>B7</th>
<th>B8</th>
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<td>1</td>
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<tr>
<td>B</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>A1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<tr>
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<td>0</td>
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<td>0</td>
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<td>0</td>
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<tr>
<td>B7</td>
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<td>0</td>
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<tr>
<td>B8</td>
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<td>0</td>
<td>0</td>
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</tr>
</tbody>
</table>

### Sub-graph relating two levels

<table>
<thead>
<tr>
<th></th>
<th>A1</th>
<th>A2</th>
<th>B6</th>
<th>B7</th>
<th>B8</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Denote $\Gamma(\mathcal{H}_2)$ where $\mathcal{H}_2$ is the sub-hierarchy consisting of just levels 1 and 2
- Graph is now directed
- Denote $\Gamma(\mathcal{H})$
Larger hierarchies (trees): labelling of nodes

- This hierarchy consists of a root (labelled $i_0 = 1$) and $q$ levels
  - This a $q$-hierarchy
- Nodes at level $m$ are labelled $i_0 i_1 \ldots i_{m-1} i_m$, $i_n = 1, 2, \ldots, n = 1, 2, \ldots, m$
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Hierarchical model: notation

- Associate a parameter vector with each node
  - Parameter vector at node $i_0 i_1 \ldots i_{m-1} i_m$ denoted $\beta_{i_0 i_1 \ldots i_m}$
  - $\beta(m) = \begin{bmatrix} 
  \beta_{i_0 i_1 \ldots i_{m-1} 1} \\
  \beta_{i_0 i_1 \ldots i_{m-1} 2} \\
  \vdots \\
  \beta_{i_0 i_1 \ldots i_{m-1} q}
\end{bmatrix}$ denotes vector of all parameters at level $m$
  - $\beta = \begin{bmatrix} 
  \beta_{(0)} \\
  \beta_{(1)} \\
  \vdots \\
  \beta_{(q)}
\end{bmatrix}$ denotes vector of all parameters at all levels
Static hierarchical model: formal statement

• Simplified version of Sundt’s (1980) model
  – A parameter vector $\beta_{i_0i_1...i_m}$ is associated with node $i_0i_1...i_m$ of the hierarchy
  – The parameter $\beta_{i_0}$ at the root of the tree is fixed
  – For $m = 0,1,...,q - 1$, the parameter vector $\beta_{i_0i_1...i_mi_{m+1}}$ is a random drawing from some distribution determined by $\beta_{i_0i_1...i_m}$
  – At each of the hierarchy’s leaves $i_0i_1...i_q$ there exists a sample of observations $y_{i_0i_1...i_qj}, j = 1,2,...$ drawn from some distribution determined by $\beta_{i_0i_1...i_q}$
Static hierarchical model: formal statement (cont’d)

– The random parameters and observations are subject to the following dependency structure:
  • \( \beta(m) = W(m-1)\beta(m-1) + \zeta(m), m = 1, \ldots, q; \)
  • \( y = X\beta(q) + \varepsilon; \)

where \( X \) is a design matrix, \( W(m-1) \) is some matrix (called a \textit{transmission matrix}) compatible with the dimensions of \( \beta(m-1) \) and \( \beta(m) \), and \( \zeta(m), \varepsilon \) are random vectors, with \( \varepsilon \) independent of the \( \zeta(m) \), and
  • \( E[\zeta] = 0, E[\varepsilon] = 0; \)
  • \( \text{Var}[\zeta] = \Lambda, \text{Var}[\varepsilon] = H; \)

where \( \zeta \) is the vector obtained by stacking the \( \zeta(m) \).
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Evolutionary hierarchical model: formal statement

- Consider a $q$-hierarchy $\mathcal{H}$, supplemented by parameters and observations that satisfy the following conditions. At each time $t = 0, 1, ...$:
  - A parameter vector $\beta^{t}_{i_{0}i_{1}...i_{m}}$ is associated with node $i_{0}i_{1}...i_{m}$ of the hierarchy.
  - For $m = 0, 1, ..., q - 1$, the parameter vector $\beta^{t}_{i_{0}i_{1}...i_{m}i_{m+1}}$ is a random drawing from some distribution determined by $\beta^{t}_{i_{0}i_{1}...i_{m}}$.
  - At each of the hierarchy’s terminal nodes $i_{0}i_{1}...i_{q}$ there exists a sample of observations $y^{t}_{i_{0}i_{1}...i_{q}j}$, $j = 1, 2, ...$ drawn from some distribution determined by $\beta^{t}_{i_{0}i_{1}...i_{q}}$. 

Evolutionary hierarchical model: formal statement (cont’d)

- The observations are subject to the following dependency on parameters:

\[ y^t = X^t \beta^t_{(q)} + \varepsilon^t, \]

where \( X^t \) is a design matrix, \( \varepsilon^t \) is a random vector, and

\[ E[\varepsilon^t] = 0, \text{Var}[\varepsilon^t] = H^t. \]

- The parameter vector \( \beta^t \) evolves over time as follows. Define \( \gamma^t_{(0)} = \beta^t_{(0)} \) and \( \gamma^t_{(m)} = \beta^t_{(m)} - W_{(m-1)} \beta^t_{(m-1)}, m = 1, \ldots, q \)
Evolutionary hierarchical model: formal statement (cont’d)

Assume that:
- The parameter vector $\beta^0$ at $t = 0$ is random with known $E[\beta^0], Var[\beta^0]$
- The parameters $\gamma_{(m)}^t$ evolve according to:
  $$\gamma_{(m)}^t = \gamma_{(m)}^{t-1} + \zeta_{(m)}^t, m = 0, \ldots, q; t = 1, 2, \ldots,$$
where $\zeta_{(m)}^t$ is a random vector, and
  $$E[\zeta^t] = 0, Var[\zeta^t] = \Lambda^t,$$
and where $\zeta^t$ is the vector obtained by stacking the $\zeta_{(m)}^t$ and $\varepsilon^t$ vectors, with $\zeta^t, \varepsilon^t, t = 0, 1, 2, \ldots$ being mutually independent.
**Alternative form of model**

**Original model**

\[ y_t = X_t \beta_{(q)} + \epsilon_t \]

**Alternative statement**

\[ y_t = U_t \gamma_t + \epsilon_t \]

where

\[ U_t = [X_t W_{(0:q)} \ X_t W_{(1:q)} \ldots \ X_t W_{(q:q)}] \]

and

\[ W_{(m:m+p)} \]

\[ = W_{(m+p-1)} W_{(m+p-2)} \ldots W_{(m)} \]
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Kalman filter (slightly simplified): model formulation

- At each $t = 0, 1, \ldots$
  - Observation (or measurement) equation
    \[ y^t = U^t \gamma^t + \varepsilon^t \]
  - System (or transition) equation
    \[ \gamma^t = \gamma^{t-1} + \zeta^t, \quad t = 1, 2, \ldots \]
- The parameter vector $\gamma^0$ at $t = 0$ is random with known $E[\gamma^0], Var[\gamma^0]$
- $\varepsilon^t \sim N(0, H^t), \zeta^t \sim N(0, \Lambda^t)$, with all $\zeta^t, \varepsilon^t, t = 0, 1, 2, \ldots$ are mutually independent
Kalman filter: parameter estimation algorithm

Let $\gamma_t|s$ and $p_t|s$ denote the estimators of $\gamma^t, \text{Var}[\gamma^t - \gamma_t|s]$ given data $\{y^0, y^1, ..., y^s\}$. The filter comprises the following procedure for each $t = 1, 2, ...$:

1) Commence with estimate $\gamma_t|t-1$ and covariance matrix $p_t|t-1$ of the same dimension

2) Calculate $F_t = U_t p_t|t-1 (U_t)^T + H_t$

3) Calculate $K_t = p_t|t-1 (U_t)^T (F_t)^{-1}$, called the Kalman gain matrix

4) Update the matrix $p_t|t-1$ as follows:
   $$p_{t+1|t} = p_t|t-1 - p_t|t-1 (U_t)^T (F_t)^{-1} U_t p_t|t-1 + \Lambda_t$$

5) Update the estimate $\gamma_t|t-1$ as follows:
   $$\gamma_{t|t} = \gamma_{t|t-1} + K_t (y_t - U_t \gamma_{t|t-1})$$

6) Further update $\gamma_t|t$ as follows:
   $$\gamma_{t+1|t} = \gamma_{t|t}$$

Credibility type of estimator
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Comparison of evolutionary hierarchical and Kalman filter models

**Evolutionary hierarchical model**
\[
y^t = U^t \gamma^t + \epsilon^t \\
\gamma_{(m)} = \gamma_{(m)}^{t-1} + \zeta_{(m)}^t
\]

**Kalman filter model**
\[
y^t = U^t \gamma^t + \epsilon^t \\
\gamma^t = \gamma^{t-1} + \zeta^t
\]

- So Kalman filter can be applied to estimate $\gamma^t$ for each $t$
  - i.e. $\beta^t$ for each $t$
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Numerical example: definition of hierarchy

- **Level 0**: Single node \{1\}
- **Level 1**: Nodes \{11, 12, 13\}
- **Level 2**: Nodes \{111, 112, 121, 122, 123, 124, 131, 132, 133, 134\}

- \(y_{i_0i_1i_2}^t\), claim frequencies per unit exposure \(E_{i_0i_1i_2}^t\) observed at the leaves
Numerical example: model structure

• Dependency of observations on parameters
  \[ y^t = X^t \beta^t_{(2)} + \varepsilon^t \quad \text{with} \quad X^t = I \]
  i.e. \( E[y^t_{i0i1i2}] = \beta^t_{i0i1i2} \)

• Dependencies between levels of hierarchy
  \( W_{(m)} = [\Gamma(H_m)]^T, \quad \text{i.e.} \quad E[\beta^t_{i0...i_{m+1}}] = \beta^t_{i0...i_m} \)

• Variances and covariances
  \( \text{Var}[y^t_{i0i1i2}] = \beta^t_{i0i1i2}/E^t_{i0i1i2} \)
  – Covariance matrices \( \Lambda^t, H^t \) diagonal
Numerical example: true parameters

<table>
<thead>
<tr>
<th>Node</th>
<th>Initial values ($\beta^0$)</th>
<th>Parameter variance ($\Lambda_t$)</th>
</tr>
</thead>
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<tr>
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<td>Mean</td>
<td>Variance</td>
</tr>
<tr>
<td>1</td>
<td>0.070</td>
<td>0.00005</td>
</tr>
<tr>
<td>11</td>
<td>0.025</td>
<td>0.00003</td>
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<tr>
<td>12</td>
<td>0.100</td>
<td>0.00030</td>
</tr>
<tr>
<td>13</td>
<td>0.150</td>
<td>0.00070</td>
</tr>
<tr>
<td>111</td>
<td>0.010</td>
<td>0.00002</td>
</tr>
<tr>
<td>112</td>
<td>0.035</td>
<td>0.00015</td>
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<tr>
<td>121</td>
<td>0.050</td>
<td>0.00040</td>
</tr>
<tr>
<td>122</td>
<td>0.080</td>
<td>0.00090</td>
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<tr>
<td>123</td>
<td>0.100</td>
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<td>132</td>
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<td>0.00500</td>
</tr>
<tr>
<td>134</td>
<td>0.200</td>
<td>0.00650</td>
</tr>
</tbody>
</table>
Numerical example: simulated data

- All nodes trendless except:
  - **Node 121**: upward trend of 0.015 per period added
  - **Node 124**: flat reduction of 0.040 made in each period
  - **Node 132**: downward trend of 0.020 per period added

<table>
<thead>
<tr>
<th>Node</th>
<th>Exposure ( (E^t_{i_0:i_1}...) )</th>
<th>Observed claim frequency at ( t = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>40</td>
<td>0.007, 0.013, 0.007</td>
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<tr>
<td>112</td>
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<tr>
<td>121</td>
<td>300</td>
<td>0.062, 0.094, 0.097</td>
</tr>
<tr>
<td>122</td>
<td>100</td>
<td>0.081, 0.088, 0.079</td>
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<tr>
<td>123</td>
<td>500</td>
<td>0.120, 0.064, 0.136</td>
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<tr>
<td>124</td>
<td>100</td>
<td>0.093, 0.053, 0.081</td>
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<td>301</td>
<td>0.150, 0.143, 0.132</td>
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<tr>
<td>132</td>
<td>50</td>
<td>0.172, 0.136, 0.093</td>
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<tr>
<td>133</td>
<td>25</td>
<td>0.111, 0.188, 0.094</td>
</tr>
<tr>
<td>134</td>
<td>20</td>
<td>0.248, 0.171, 0.195</td>
</tr>
</tbody>
</table>
Numerical example: results (1)

- Node 121: increasing trend in true frequency
Numerical example: results (2)

- Node 124: one-off downward shift in true frequency at $t = 1$
Numerical example: results (3)

- Node 132: decreasing trend in true frequency
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Conclusion

• Evolutionary hierarchical model has been formulated
• Parameter estimates obtained by application of the Kalman filter
  – Filter updates estimates from one epoch to the next as further data are observed
• Application of the Kalman filter conceptually straightforward, but
  – Tree structure of the model parameters can be extensive
  – Some effort is required to retain organization of the updating algorithm
  – Achieved by suitable manipulation of the adjacency matrix associated with the tree
  – Adjacency matrix then recruited to play its role in Kalman filter matrix calculations
References


Questions?