Fitting and Estimating Risk Dependence using Copulas for Multivariate Data

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Introduction

- Standard market risk and portfolio modelling assumption is linear dependence (multi-variate normal distributions)
- Correlation or covariance - usually used to measure "dependence" especially in asset portfolios
- Emphasis on expected values, variances, tails and less on dependence of risks in portfolios
- Law of Large numbers works for averages but for total risk exposure dependence is important (especially in the tails)

Introduction

- RiskMetrics and market risk models do not allow for higher levels of tail dependency observed

- Credit risk models often ignore dependence between loss frequency and loss amount

- Credit risk models often used portfolio risk assumption is multivariate normal distribution (Gaussian copula)

- Financial markets have higher correlation when markets fall

Introduction

- Copulas are now a major tool in modelling the dependence structure of risks in insurance and finance
- Major issue is how to develop and fit copulas that match market data better than the Gaussian copula
- A technique is proposed that is effective and readily implemented for practical applications for internal risk based capital models.

Copulas

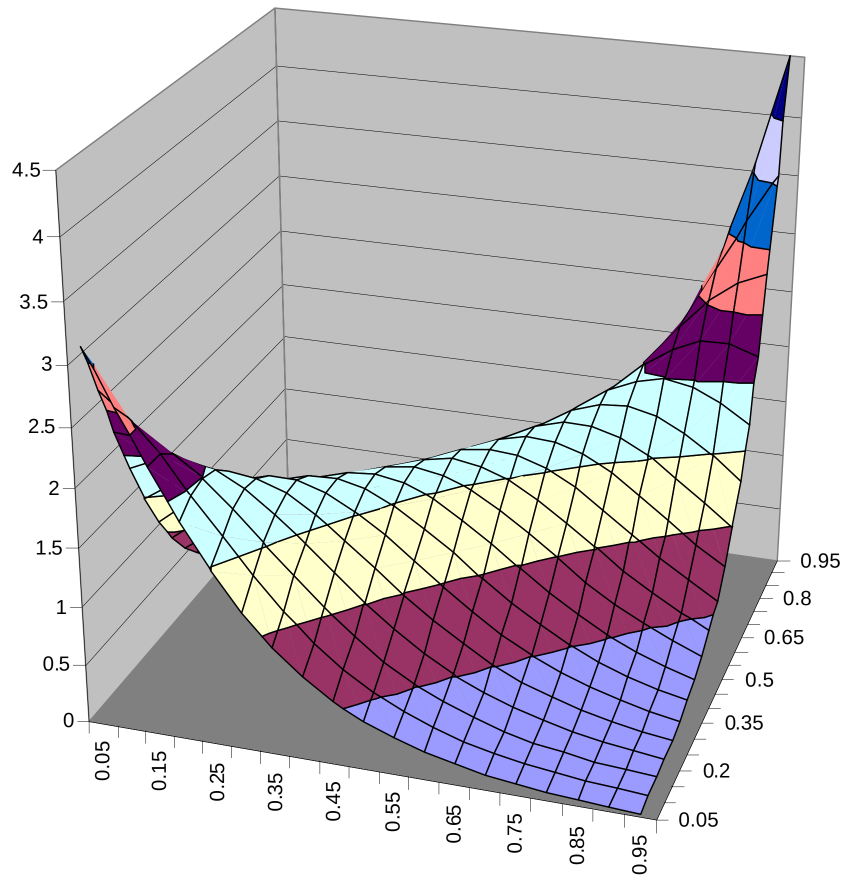
- Joint probability distribution (assume d risks with continuous strictly increasing distribution functions)

$$F_X(x_1, \ldots, x_d) = \Pr(X_1 \leq x_1, \ldots, X_d \leq x_d)$$

- Usually have data on marginal distributions of each risk (or we assume a distribution) $F_{X_1}, \ldots, F_{X_d}$ where $F_{X_i}(x) = \Pr(X_i \leq x_i)$

- A Copula is a function that can allow modelling of (non-linear) dependence between marginal distributions

Copula Density - Example



Copulas

Sklar (1959) noted that a joint distribution function $H = H(x, y)$ could be expressed in terms of its marginals $F = F(x)$ and $G = G(y)$ by

$$H(x, y) = C(F(x), F(y))$$

for a suitable function $C : [0, 1] \times [0, 1] \to [0, 1]$, and that this function C (called a **copula**) is unique if F and G are continuous.

See: Schweizer and Sklar (1983/2005) Chapter 6 or Nelsen (1998/2005) Chapter 2.

Copulas

Axioms for 2-Copulas

$C : [0,1] \times [0,1] \to [0,1]$ satisfies:

1. $C(u,0) = 0 = C(0,v)$ if $u, v \in [0,1]$.

2. $C(u,1) = u$ and $C(1,v) = v$ if $u, v \in [0,1]$.

3. For all $u_1 \le u_2$ and $v_1 \le v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0$$

C is **2-monotone** if it satisfies (3).

Archimedean Copulas

$$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v))$$

where $\phi : [0,1] \to [0,\infty]$ is continuous, strictly decreasing with $\phi(1) = 0$ and

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & \text{if } 0 \le t \le \phi(0) \\ 0 & \text{if } \phi(0) \le t \le \infty \end{cases}$$

Such a C satisfies axioms 1 and 2 and is a copula if and only if ϕ is convex.

We often have $\phi(0+) = \infty$ and then $\phi^{[-1]} \equiv \phi^{-1}$.

ϕ is called the **additive generator** of C.

Multiplicative Generator

We can also write an Archimedean copula using a multiplicative generator ψ as follows:

$$C(u,v) = \psi^{[-1]}\left(\psi(u)\psi(v)\right)$$

where for $t \in [0,1]$

$$\psi(t) = \exp(-\phi(t))$$

Where we now require $\psi : [0,1] \to [0,1]$ be continuous, strictly increasing, $\psi(1) = 1$ and $t \to -\log(\psi(t))$ be convex. We usually have $\psi(0) = 0$ so that $\psi^{[-1]} \equiv \psi^{-1}$. We note that ψ concave is a sufficient condition, but not a necessary condition for C to be a copula. If $\phi(t) = (-\log(t))^{\theta}$ with $\theta > 1$ (Gumbel copula, see Nelsen page 118) then ψ is not concave on $[0,1]$.

Distortion of Copulas

If C is a 2-copula, then we can form C^{ψ} the **distortion** of C by ψ by setting

$$C^{\psi}(u,v) = \psi^{-1}\left(C(\psi(u),\psi(v))\right)$$

where $\psi : [0,1] \to [0,1]$ is continuous, strictly increasing and concave with $\psi(0) = 0$ and $\psi(1) = 1$. Under these conditions C^{ψ} will again be a copula. The concavity of ψ (and suitable generalizations for n-copulas) are sufficient for C^{ψ} to be a copula, but this requirement could be weakened. These transformation were introduced by Genest (2000).

Archimedean copulas are then distortions of the **product (independence) copula**.

Gaussian Copula

The Gaussian (or Normal) 2-copula has the form

$$C(u,v) = \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v), \varrho\right)$$

where $\Phi_2(\cdot,\cdot,\varrho)$ denotes the bivariate distribution function with correlation parameter ϱ, Φ is the standard normal distribution function. It can be rewritten

$$C(u,v) = \int_0^u \Phi\left[\frac{\Phi^{-1}(v) - \varrho\ \Phi^{-1}(z)}{\sqrt{1-\varrho^2}}\right] dz$$

We will be interested in the distortion of such Gaussian 2-copulas.

It generalizes the Archimedean copula construction.

Bernstein Copula

Given a copula C we can for each integer $n \geq 1$ construct a **Bernstein copula** from it as follows:

$$B_n(C)(u,v) = \sum_{i,j=1}^{n} C\left(\frac{i}{n}, \frac{j}{n}\right) p_{n,i}(u) p_{n,j}(v)$$

where

$$p_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

The Bernstein copula is a polynomial in u and v and

$$|B_n(C)(u,v) - C(u,v)| \leq \frac{5}{2\sqrt{n}}$$

for all $u, v \in [0,1]$.

Example: FGM

For example, if $C(u,v) \equiv uv[1+\theta(1-u)(1-v)]$ with $|\theta| \leq 1$ (FGM copula), then

$$B_n(C)(u,v) = uv + \left(\frac{n-1}{n}\right)^2 \theta u(1-u)v(1-v)$$

Empirical Copulas

Let $\{(x_k, y_k)\}_{k=1}^{m}$ denote a sample of size m from a continuous bivariate distribution. The **empirical copula** is a function EC_m given by

$$EC_m\left(\frac{i}{m}, \frac{j}{m}\right) = \frac{\#\{(x, y) \mid x \leq x_{(i)},\ y \leq y_{(j)}\}}{m}$$

where the pairs (x, y) counted are from the sample and where $x_{(i)}$ and $y_{(j)}$ for $1 \leq i, j \leq m$ denote order statistics from the sample.

To smooth the empirical copula, we can form (for n a divisor of m) a **Bernstein empirical copula** $B_n(EC_m)$ (for example $m = 100$ and $n = 10$).

Distortion Functions

Given that D is a copula so that $D = C^{\psi}$ we would like to determine the function ψ.

One approach is to solve the **diagonal equation**:

$$D(u,u) = C^{\psi}(u,u)$$

which is the same as

$$\psi(g(u)) = f(\psi(u))$$

where $g(u) \equiv D(u,u)$ and $f(u) = C(u,u)$.

Distortion Functions

Even if $D = C^{\psi}$ were to hold approximately we would still like to find ψ. If D were an empirical Bernstein copula we would have $f(u) = u^2$ if we were fitting an Archimedean copula, and we would have $f(u) = C_{\rho}(u, u)$ if we were trying to fit a distorted Gaussian copula to some data represented by this empirical Bernstein copula.

Diagonal Equation

The diagonal equation is:

$$\psi(g(u)) = f(\psi(u))$$

We now study its solution. We make the following assumptions which are motivated by important examples.

Diagonal Equation - Assumptions

1. f and g are increasing functions of $[0,1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$.

2. $g(u) < u$ and $f(u) < u$ for $0 < u < 1$.

3. $f'(0) = g'(0) = 0$ and $g''(0) > 0$

4. f is strictly convex on $[0,1]$.

Important Result (Lemma)

Lemma:

Suppose for all $0 \leq u \leq 1$

$$\psi(u) = \lim_{n \to \infty} f^{(n)} \circ g^{(-n)}(u)$$

exists. Then this ψ satisfies the diagonal equation.

Here we have defined:

$f^{(n)} = f \circ f \circ ... \circ f$ (n-fold composition) and $g^{(-n)} = g^{-1} \circ g^{-1} \circ ... \circ g^{-1}$ (n-fold composition).

Numerical Algorithm

We construct an approximate solution for ψ as follows:

For some positive integer n we set

$$u_j := \frac{j}{n} \quad \text{for} \quad j = 0, 1, ..., n$$

and seek ψ piecewise linear with

$$\psi_j \approx \psi(u_j)$$

so that (recall that $g(u_j) < u_j$)

$$\psi(g(u_j)) = f(\psi_j)$$

with $\psi(u_0) = \psi(0) = 0$ and $\psi(u_n) = \psi(1) = 1$.

Algorithm

We first assign a value to ψ_1. If we can find a-priori an expression for $\psi'(0)$ then we may set $\psi_1 = \psi'(0)/n$. We have the Lemma:

Lemma: If $\psi'(0)$ and $\psi''(0)$ are finite, then

1. If $f(u) = u^2$, then $\psi'(0) = \frac{1}{2}g''(0)$

2. If $f(u) = u^2(1 + \theta(1-u)^2)$ then $\psi'(0) = \frac{1}{2(1+\theta)}g''(0)$

Gaussian Copula

A Special Case : $f(u) = C_\rho(u, u)$

We have the asymptotic results as $u \to 0+$.

$$\Phi^{-1}(u) \sim -\sqrt{-2 \ln u}$$

$$f(u) \sim \frac{(1+\rho)^2}{\sqrt{1-\rho^2}} u^{\frac{2}{1+\rho}}$$

$$g(u) \sim \frac{1}{2} g''(0) u^2$$

and so from

$$\psi(g(u)) = f(\psi(u))$$

Asymptotic Equation

The asymptotic equation can be derived

$$\psi\left(\frac{1}{2}g''(0)u^2\right) = \frac{(1+\rho)^2}{\sqrt{1-\rho^2}}\left[\psi(u)\right]^{\frac{2}{1+\rho}}$$

Asymptotic Equation Solution

The solution of asymptotic equation is:

$$\psi(u) = \phi\left(\frac{1}{2} g''(0) u\right)$$

with

$$\phi(v) = \exp\left(-\frac{b}{a-1} - \gamma(-\ln v)^{\lambda}\right)$$

where

$$a = \frac{2}{\rho + 1}$$

$$b = \ln\left[\frac{(1+\rho)^2}{\sqrt{1-\rho^2}}\right]$$

$$\lambda = \frac{\ln a}{\ln 2} = 1 - \frac{\ln(1+\rho)}{\ln 2}$$

$\gamma \in (0, 1)$ is arbitrary

So we can use $\psi_1 = \phi(ng(1/n))$.

Algorithm – Step 2

Assume that ψ_j is known for $j = 1, 2, ..., k$ with $k < n$ and is a strictly increasing sequence of values in $(0, 1)$. Then $\psi^{|k}$ can be specified on $[0, 1]$ as follows:

$$\psi^{|k}(u) = \sum_{j=0}^{k-1} \alpha_j (u - u_j)^+$$

where

$$\alpha_0 = n(\psi_1 - \psi_0) = n\psi_1$$
$$\alpha_j = n(\psi_{j+1} - 2\psi_j + \psi_{j-1})$$

for

$$j = 1, 2, ..., k - 1.$$

Algorithm – Step 3

We now compute $\psi_{k+1} > \psi_k$ (where $k+1 < n$).

There are two cases:

(1) If $g(u_{k+1}) \le u_k$ then solve

$$f(\psi_{k+1}) = \psi^{|k}(g(u_{k+1}))$$

(2) If $u_k < g(u_{k+1}) < u_{k+1}$ then solve

$$\begin{aligned} f(\psi_{k+1}) &= \psi_k + n\left[g(u_{k+1}) - u_k\right]\left[\psi_{k+1} - \psi_k\right] \\ &\equiv \psi^{|k+1}(g(u_{k+1})) \end{aligned}$$

Step 3 – Case 1

$$h(z) = f(z) - \psi^{|k}(g(u_{k+1}))$$

then

$$\begin{aligned} h(\psi_k) &= f(\psi_k) - \psi^{|k}(g(u_{k+1})) \\ &= \psi^{|k}(g(u_k)) - \psi^{|k}(g(u_{k+1})) < 0 \end{aligned}$$

since $g(u_k) < g(u_{k+1})$ and $\psi^{|k}$ is strictly increasing

$$\begin{aligned} h(1) &= 1 - \psi^{|k}(g(u_{k+1})) \\ &\geq 1 - \psi^{|k}(u_k) = 1 - \psi_k > 0 \end{aligned}$$

and so there is a solution $\psi_k < z < 1$ of $h(z) = 0$, and this solution is unique as $h'(u) = f'(u) > 0$ for $0 < u < 1$. We set ψ_{k+1} to be this unique solution and we have $\psi_k < \psi_{k+1} < 1$.

Step 3 – Case 2

We set

$$h(z) = f(z) - \psi_k - n[g(u_{k+1}) - u_k][z - \psi_k]$$

then

$$\begin{aligned} h(\psi_k) &= f(\psi_k) - \psi_k < 0 \\ h(1) &= n(1 - \psi_k)(u_{k+1} - g(u_{k+1})) > 0 \end{aligned}$$

and so there is a solution $\psi_k < z < 1$ of $h(z) = 0$, and this solution is unique since h is strictly convex on $(0, 1)$ as $h''(u) = f''(u) > 0$ for $0 < u < 1$. We set ψ_{k+1} to be this unique solution and we have $\psi_k < \psi_{k+1} < 1$.

Algorithm - Step 4

Let us also note the recurrence:

$$\psi^{|k+1}(u) = \psi^{|k}(u) + \alpha_k (u - u_k)^+$$
$$\psi(u) = \psi^{|k}(u) \quad \text{on} \quad [0, u_k]$$

an in particular

$$\psi(u) \equiv \psi^{|n}(u) \quad \text{on} \quad [0, 1]$$

Example 1

$$f(u) = u^2$$

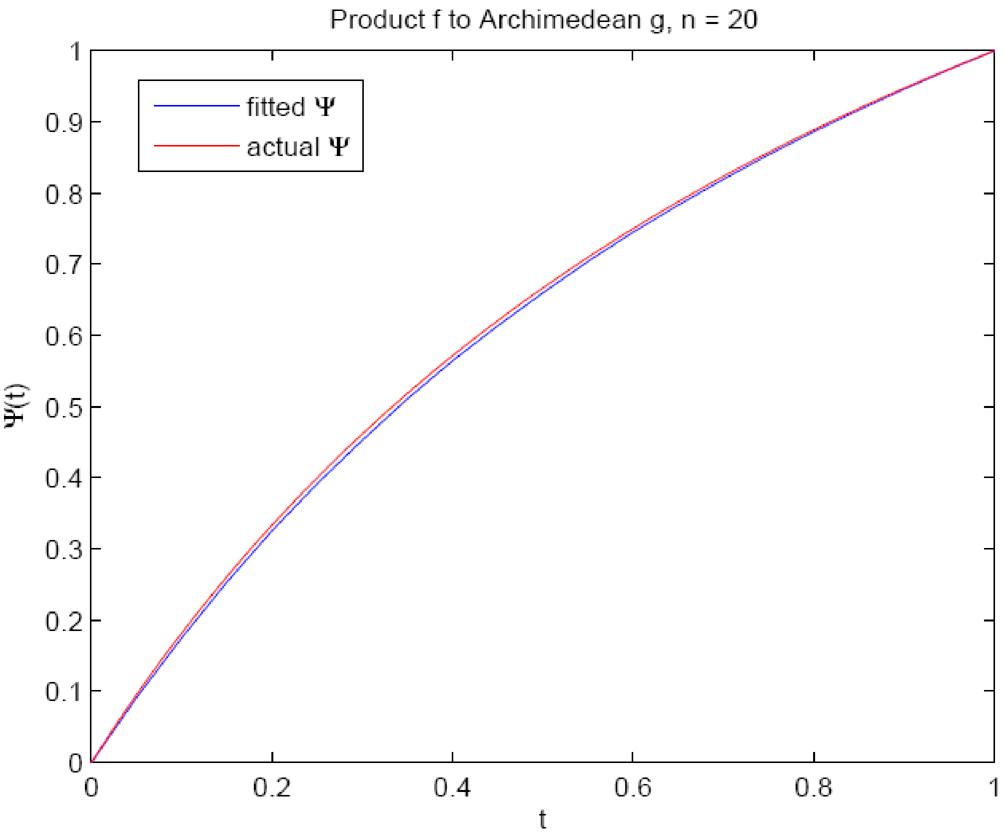
$$g(u) = \frac{u^2}{1 - \alpha(1-u)^2}$$

g is the diagonal of an Archimedean copula with multiplicative generator

$$\psi(t) = \frac{t}{1 - \alpha(1-t)}$$

$\alpha = 0.5$ and $n = 20$ were used. The algorithm correctly reconstructs the multiplicative generator (distortion function of the product copula).

Example 1


Product f to Archimedean g, n = 20
fitted Ψ
actual Ψ
Ψ(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

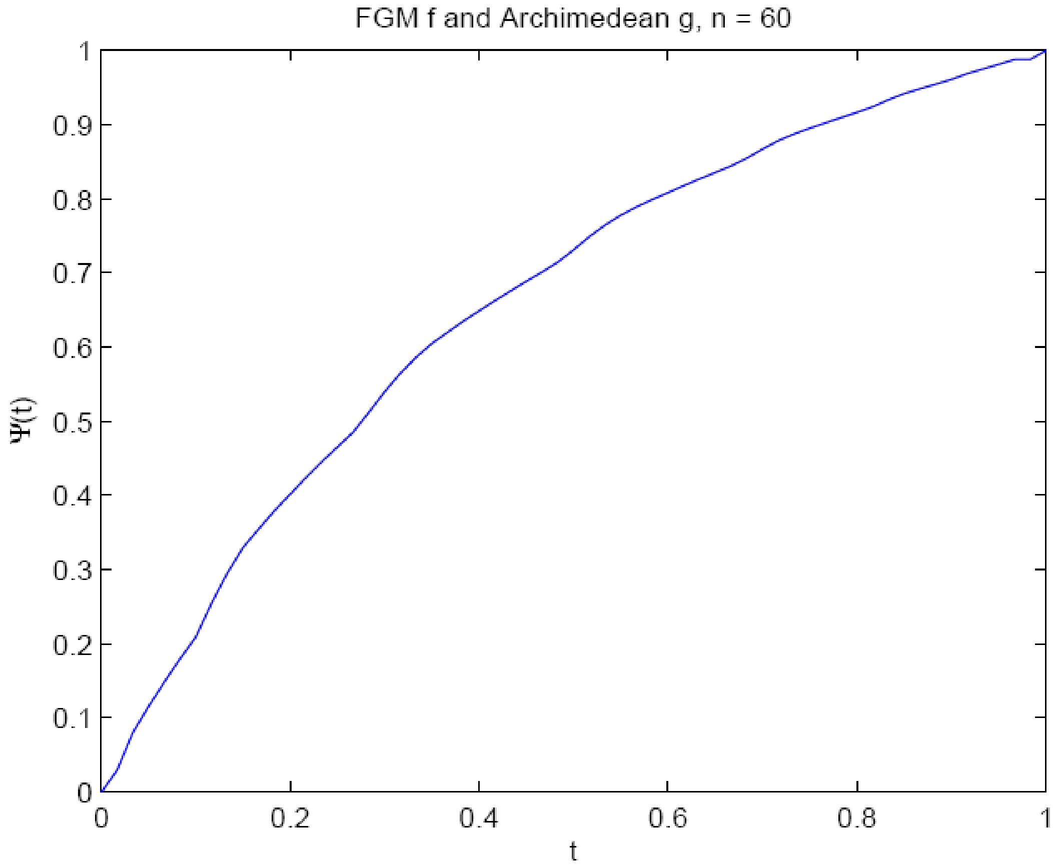
Example 2

$$f(u) = u^2 + \theta u^2 (1-u)^2$$
$$g(u) = \frac{u^2}{1 - \alpha(1-u)^2}$$

f is the diagonal of an FGM copula and g is the diagonal of an Archimedean copula as in Example 1. Here f is not an Archimedean copula.

$\theta = -0.5$, $\alpha = 0.5$ and $n = 60$ were used.

Example 2


FGM f and Archimedean g, n = 60
Ψ(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

Example 3

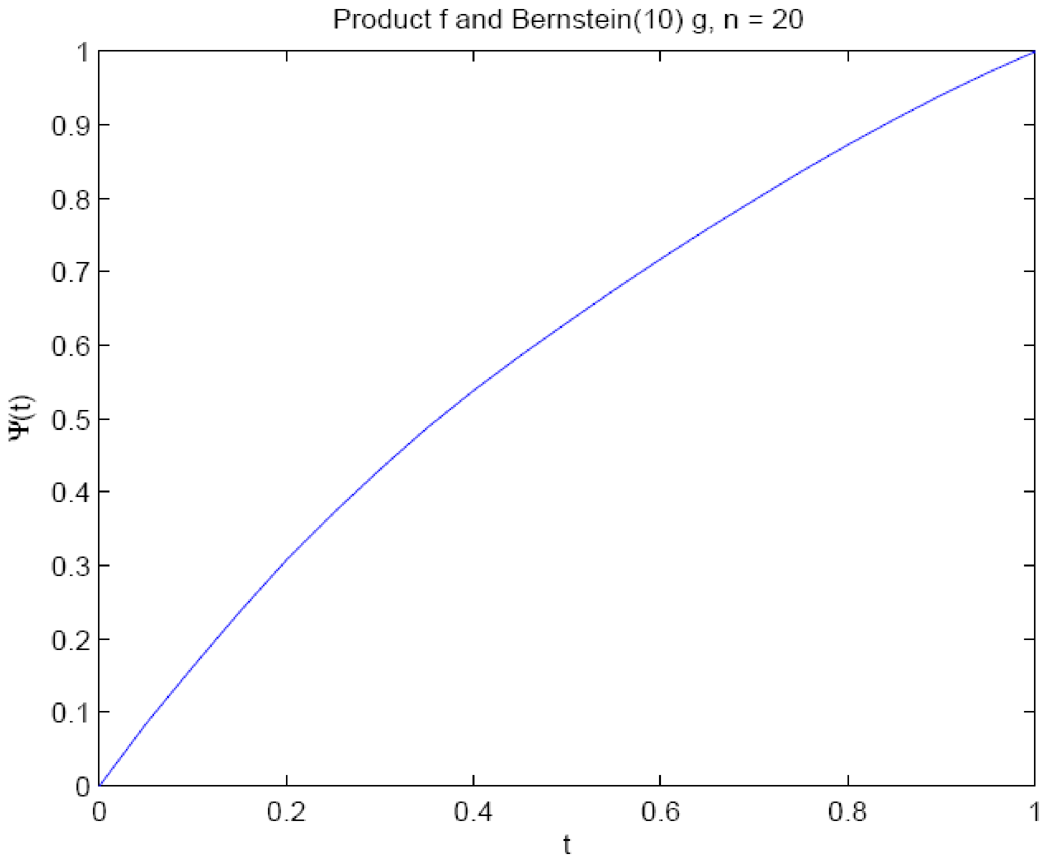
$$f(u) = u^2$$
$$g(u) = B_{10}(C)(u,u)$$

f is the diagonal of the independence/product copula and g is the diagonal of an empirical Bernstein copula.

The empirical data came from 100 draws of (X,Y) from a joint distribution H with $H(x,y) = C(F(x), G(y))$ where F and G are the standard normal distribution functions, and C is the FGM copula $C(u,v) \equiv uv + \theta u(1-u)v(1-v)$ and $\theta = 0.1\pi$ was chosen.

$n = 20$ was used.

Example 3


Product f and Bernstein(10) g, n = 20
Ψ(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

Example 4

$$f(u) = C_\rho(u, u)$$
$$g(u) = B_{10}(C)(u, u)$$

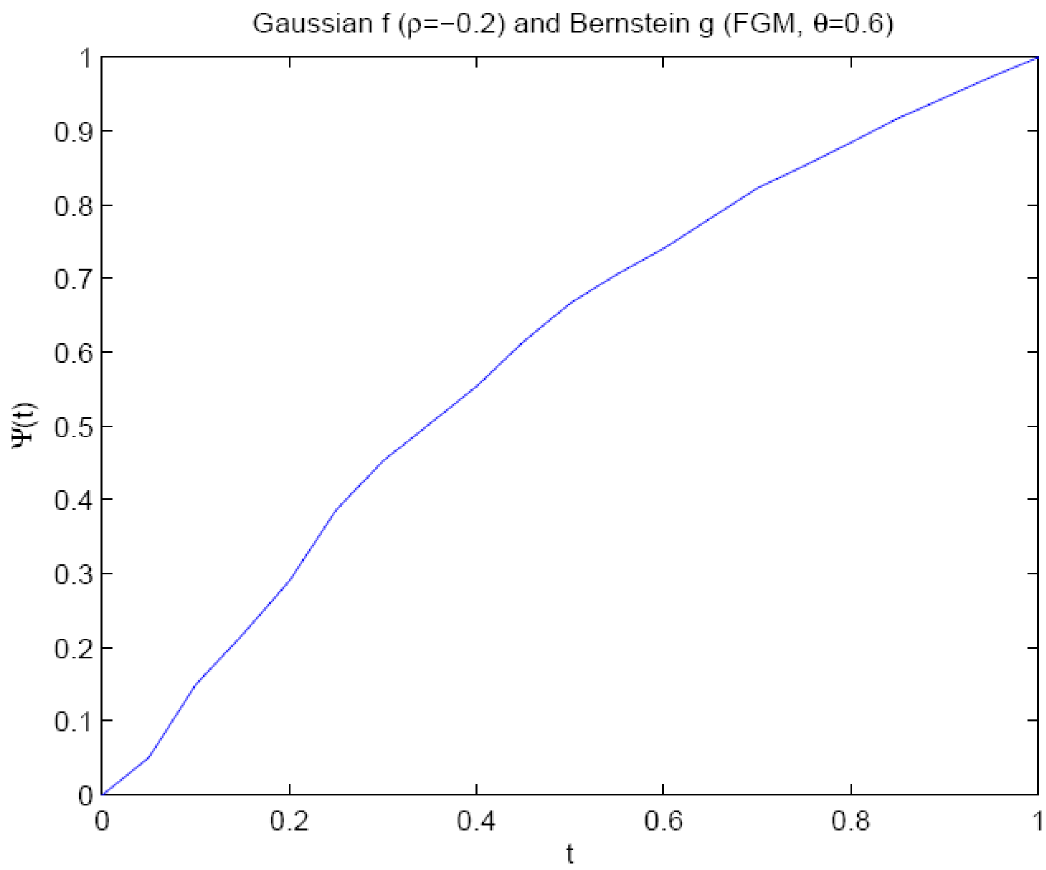
f is the diagonal of the Gaussian copula and g is the diagonal of an empirical Bernstein copula.

$\rho = -0.2$ was selected in f.

The empirical data came from 100 draws of (X, Y) from a joint distribution H with $H(x, y) = C(F(x), G(y))$ where F and G are the standard uniform distribution functions, and C is the FGM copula $C(u, v) \equiv uv + \theta u(1 - u)v(1 - v)$ and $\theta = 0.6$ was chosen.

$n = 20$ was used.

Example 4


Gaussian f (ρ=−0.2) and Bernstein g (FGM, θ=0.6)
Ψ(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

Summary

- This talk has presented a new approach to fitting copulas to empirical data
- This has been implemented with a numerical algorithm
- The algorithm performs well with examples of common copulas and has widespread practical application

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