Continuous Compounding, Volatility and Beta

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Motivation

- Clarify when to use different mean rates of return and the definition of return to use for CAPM
- Recent paper by Fitzherbert (2001) and the Discussion (AAJ, Volume 7, Issue 4, 681-714, 715-754) demonstrate
  - misconceptions about empirical studies and assumptions of CAPM and
  - errors in the use of average returns

Investment Objectives

- Fitzherbert (2001) Summary and Conclusions
  - “mean return should be defined as the mean of continuously compounded return or its equivalent when making investment decisions on the basis of ‘maximising their terminal wealth’.”
- CAPM does not assume investors make decisions based on maximising terminal wealth

Investment Objectives

- CAPM - single period mean-variance of wealth maximisers (NOT Terminal wealth but risk adjusted terminal wealth)
  - NOT W (a random variable) but $E[W]-\lambda Var[W]$
  - Note $E[W]=W(0)[1+E[R]]$
  - Need to take risk into account and investment decisions are not based on "terminal wealth" but characteristics of the distribution of terminal wealth such as mean, variance or other risk measures

Investment Objectives

- Can extend to a multi-period model
  - $W(T)=W(0)[1+R(1)]\ldots.[1+R(T)]$ where the returns are random variables
  - Can still maximise risk-adjusted terminal wealth $E[W(T)]- \lambda Var[W(T)]$ and note that if independent
    - $E[W(T)]=W(0)[1+E[R(1)]]\ldots.[1+E[R(T)]]$
  - Assuming that returns are identically distributed then
    - $E[R(1)]= E[R(2)]=\ldots.=E[R(T)]=E[R]$
    - $E[W(T)]=W(0)[1+E[R]]^T$

Investment Objectives

- What if we have historical data to estimate returns?
  - Require an estimate of E[R] when you have a sample of returns r(1), r(2), …, r(s)
  - Since identically distributed these can be treated as a simple random sample from the distribution of R
  - MLE (and least squares estimate) for E[R] is sample mean (arithmetic average) of r(1), r(2), …, r(s)

Investment Objectives

- What if we use continuous compounding returns $\delta=\ln[1+R]$?
  - $W(T)=W(0)[\exp(\delta(1))]....[\exp(\delta(T))]$ where the returns are random variables
  - Can still maximise risk-adjusted terminal wealth $E[W(T)]- \lambda Var[W(T)]$ and note that
    - $E[W(T)]=W(0)E[\exp(\delta(1)+\delta(2)+.... \delta(T))]$
    - For convenience, often assume $\delta(s)$ are independent normally distributed with mean μ and variance σ^2 in which case:

$$E[\exp(\delta(1)+\delta(2)+.... \delta(T))]=\{\exp[\mu+1/2\sigma^2]\}^T$$

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Investment Objectives

- What if returns are not independent?
  – This is studied in Subject 103 of the Institute of Actuaries syllabus
  – Time series and econometric models include allowance for dependence - autoregressive, moving average, volatility dependence (GARCH)
  – Need to estimate parameters of the model using maximum likelihood
  – See course notes for Subject 103

Misleading Means of Discrete Rates of Return

- Table 2 Fitzherbert
  - Example compares a fixed per period investment with a variable return investment
  - The variable return must be a sample path from a possible distribution (sample of 2 to estimate a mean return!)
  - These two cases are not comparable - need to identify the distribution of returns that the second case is taken from
  - Here is a valid comparison

Misleading Means of Discrete Rates of Return

- **Table 2 Fitzherbert**

Time	0	1	2
			400
		200	
			40
Value	100		
			40
		20	
			4
Average portfolio value		110	121
Average return % p.a.		10%	10%

Misleading Means of Discrete Rates of Return

- Table 2 Fitzherbert
  - Need to allow for the fact that these are sample paths of returns
  - Estimation of the expected value of a return distribution is different to summarising the equivalent annual average return along a sample path for two different investments with different risks and returns

Arithmetic and Geometric Mean Rates of Return

- **Approximate relationship**
  - geometric average = arithmetic average minus one half variance
  - log-normal
    - $E[R]=\exp(\mu+1/2\sigma^2)-1$ where $\mu=E[\delta]$ and σ^2 is variance of δ
    - μ and σ are not the sample estimates
  - note not a geometric average of returns (geometric average of 1+r)
  - Details in our Convention paper

Continuous Compounding

- Fitzherbert (2001) Summary and Conclusions
  - any model of investment returns needs to establish a relationship between the model's variables and the mean continuously compounded return

Continuous Compounding

- Continuous time CAPM does exactly that (Merton, 1970 Working Paper)
  - $E[\delta(i)]=r+ (\sigma_{iM}/\sigma_M^2)[E [\delta(M)-r]+1/2(\sigma_{iM}-\sigma_i^2)$
  - this is multi-period (and applies for a single period in a multi-period model)
  - studies referred to by Fitzherbert that use continuous compounding to test CAPM USE THIS FORM OF THE MODEL (Jensen, Basu)
  - Details in our Convention paper (Section 5.1)

CAPM Tests - BJS (1972)

- **Fitzherbert (2001) Summary and Conclusions**
  - **"..most of the empirical academic research supporting a positive linear relationship between β and mean return has been based on arithmetic means of discrete rates of return such as Black, Jensen and Scholes (1972).."**

CAPM Tests - Jensen (1972) and BJS (1972)

- Jensen (1972), BJS 1972
  - Regardless of whether or not discrete compounding or continuous compounding is used, the positive relationship between expected return and beta holds in these studies (see Section 6.1 of our paper)

Confusion reigns

- Fitzherbert (2001) Section 3.3
  - "Consequently, when an investor is making decisions on the basis of mean rates of return, *the only definition of 'mean return' that makes any sense is mean continuously compounded return or something that is equivalent*."

Confusion reigns

– Difference between COMPOUNDING FREQUENCY (per annum, continuously) and AVERAGING (arithmetic, geometric)

– *"mean continuously compounded return or something that is equivalent"???*

  - mean = arithmetic average or geometric average?
  - what is equivalent?

– It is important (even for actuaries) to explain and communicate the financial maths clearly

CAPM

- Our paper is NOT about the validity of the CAPM nor the results from early studies (these results look fine to us)
- CAPM is a MODEL
  - all models are wrong and some are useful (and some should only be used by those who understand what they are doing)
- CAPM empirical evidence is mixed but the assumptions are simplistic - more realistic models are often required

CAPM

- Many other models of pricing/expected returns developed based on empirical tests and more recent theoretical developments
  - APT (multiple factors)
  - Consumption based CAPM
  - Models of returns allowing for taxes, transaction costs etc
  - OPT (and martingale pricing)
  - Incomplete markets (actuarial pricing)
  - Real options

CAPM

- Need to understand the application and select the appropriate model
- Different issues and models required for
  - Project finance (discounting expected cash flows)
  - Cost of capital (investment decisions, other factors including tax, options to defer)
  - Tactical asset allocation (multiple factors, dynamic models)
  - Fair rate of return in insurance (incomplete markets, market frictions)

CAPM - beta

- Consider two investments with the <u>same expected future cashflow</u> (retained earnings and dividends) that form a small part of your total wealth, and assume you hold a well diversified portfolio
- Investment A - if the value of the well diversified portfolio goes <u>up</u>, then its value goes <u>up</u> and if the value of the well diversified portfolio goes down, then its value goes down
- Investment B - if the value of the well diversified portfolio goes <u>down</u>, then its value goes <u>up</u> and if the value of the well diversified portfolio goes up, then its value goes down
- Would you pay more for A or B?

CAPM - beta (hint)

Payoffs						
	A	**B**	**Wealth**	**Wealth+A**	**Wealth+B**	**Probs**
	110	90	2000	2110	2090	0.5
	90	110	1500	1590	1610	0.5
Expected	100	100	1750	1850	1850	
Variance	100	100	62500	67600	57600	
SD	10	10	250	260	240	

Actuarial Education

- Part I
  - should exploit the actuarial syllabus to ensure students have a general understanding of valuation and risk management models (not just CAPM, APT, OPT as in current syllabus)
  - emphasis on applications of interest to actuaries and to give them an advantage over finance students

Actuarial Education

- Part II
  - links to practice
  - basic applications of models and understanding their shortcomings
- Part III
  - more advanced coverage across the practice areas (not just in investment and finance subjects)
  - recent developments in models for asset pricing and actuarial and related applications in risk management (real options, incomplete markets, frictional costs) - beyond basic finance theory

Conclusions

- The Covention paper is NOT about the CAPM
- It is about
  - the correct use of mean returns and
  - interpretation of results in a number of published studies
- Arithmetic averages, assuming independent indentically distributed returns, should be used for projecting <u>expected</u> future values (<u>should normally consider the full distribution</u>)
- Arithmetic averages of per period returns are the correct statistical estimates for CAPM expected returns based on historical data and assumption of independent, identically distributed returns

Conclusions

- Comparing different investments from historical data use IRR, Time-weighted returns (allowing for cashflows)
  - for an historical sample of returns with NO CASHFLOWS the geometric average summarises the sample path into a single per period equivalent return (but ignores risk)
- No need to worry about handing back Nobel prizes (CAPM derivation is correct)