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Living Life Optimally with a Mean-Reverting Price of Risk*

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Abstract

With life expectancy ever increasing in this modern age, financial planning has never been a more important issue for individuals. What should one's consumption pattern be and what level of investment should one make in order to maximise utility over one's lifetime? This paper presents an economic model which quantitatively solves these issues in a complex environment where returns are mean-reverting and correlated with the risky asset. To further reflect the real world we incorporate event risk, represented by sudden changes in the value of the risky asset. We analyse the interaction between the consumption level and the level of investment for individuals over time. In addition, we observe how individuals' insurance behaviour changes over the course of their lifetimes.

Keywords: Financial planning; utility theory; consumption; event risk; stochastic optimal control.

Journal of Economic Literature Classification Numbers: C61, D14, D91, G11, G22, H55, J26.

1 Introduction

Merton is regarded as a pioneer in applying stochastic optimal control to problems in finance. His seminal paper in 1969 and subsequent extension (Merton, 1971), on applying dynamic stochastic techniques to the optimal lifetime consumption and investment problem has captured the attention of many researchers. For example, Richard (1975) was one of the earlier academics to adapt the Merton model to include mortality and insurance. Over the years, researchers have gradually extended the traditional model to incorporate more richer stochastic environment. Some, but not limited to, include Heston (1992) on stochastic volatility, Kim and Omberg (1996) on mean-reverting risk premium and Henderson (2004) on stochastic income.

This paper will focus on one of these areas, namely mean-reverting returns. We aim to study the impact of jumps on optimal consumption and investment strategy for a finitely-lived investor when the risk premium is mean-reverting and predictable.

The reason for choosing this area is that there is empirical evidence suggesting investment returns are mean reverting. Poterba and Summers (1988) show that serial negative correlation seems to exists in the longer horizon over the US and 17 other countries, suggesting mean reversion. The inclusion of jumps in the risky asset is motivated by recent papers on event risk[1] as well as it being a more realistic model. Finally, consumption is included since it is an important aspect of investor's behaviour. Part of the reason for investment is to fund consumption.

Before we enter into the details of our problem, it is necessary to first define what it is meant by an optimal consumption and investment strategy. We will start off by defining an optimal investment strategy, as it is simpler to understand. By optimal investment strategy, we refer to an agent choosing an investment strategy such that it maximises some objective function, such as maximising the expected terminal wealth over a period of time. In a simple two-asset model, the investment strategy involves choosing the proportion of wealth allocated to both the risky asset and the risk-free asset. Thus, the agent has to choose between the risk-free asset which has a low return, and the risky asset which has a higher return but simultaneously entails more risk.

By the same token, optimal consumption is defined as maximising the expected utility derived from consuming his/her wealth over the period.[2] This

[1] Liu, Longstaff and Pan (2003) investigated the optimal control problem for stochastic volatility in a double jump model developed by Duffie, Pan and Singleton (2002).

[2] The optimal investment strategy problem is actually a subset of this problem as one could assume that the agent consumes all his/her wealth at the end of the period.

problem is now slightly more complex as the agent is now required to make two decisions. Not only does he/she need to formulate an investment strategy, he/she also has to determine how to consume his/her wealth over time to obtain the highest expected utility. These two decisions are often interlinked since consumption is based on the current wealth which is determined by the investment strategy taken.

In this work, we wish to observe how jumps and mean-reversion with predictability affect these dynamics.

The paper is organised as follows. Section 2 reviews some optimal control papers relating to mean-reversion. Section 3 extends the Wu (2003) and Wachter (2002) models. Section 4 explains the numerical technique that was used to solve the stochastic optimal control we are dealing with. Section 5 reports the results of our modelling and analyses the findings while section 6 concludes.

2 Literature Review

This section will examine three key aspects of mean-reversion by critically review three papers. Other references are made throughout to complement these studies. We will begin with an analysis of the plain-vanilla mean-reverting model by Kim and Omberg. The objective of this is to demonstrate that predictability has an important impact on the optimal allocation. More specifically, the concept of 'intertemporal demand' (a form of non-myopic demand) is introduced to show that investors no longer consider the problem as a single period problem. Subsequently, we will discuss the impacts of jumps on the Kim and Omberg model (Wu, 2003). This is of significance as empirical studies have shown that diffusion processes do not provide good fit to 'financial' data (Gallant and Tauchen, 1997). Finally, the optimal consumption problem in this environment, developed by Wachter (2002), is reviewed. We will establish that consumption, which has recently been overlooked by investigators in this area, does have an explicit influence on the optimal allocation strategy.

2.1 Kim and Omberg

Kim and Omberg investigated the optimal allocation problem in an environment where the risk premium is mean-reverting as well as being correlated with the risky asset.[3] Mathematically, the risky asset follows the geometric

[3]Merton (1971) had investigated a very similar problem. In his model, it is the return of the asset that is mean reverting. Also, the two processes (risky asset and risk premium)

Brownian process

$$\frac{dS}{S} = (\alpha_t)dt + \sigma dZ_1, \tag{1}$$

where α_t is the instantaneous expected return, σ^2 is the stock's volatility, Z_1 is a standard, one dimensional Brownian motion. The risk premium,X_t, also known as the Sharpe ratio, is defined as

$$X_t = \frac{\alpha_t - r}{\sigma} \tag{2}$$

where r represents the risk-free rate and obeys a mean-reverting Ornstein-Uhlenbeck process

$$X_t = -\lambda_x(X_t - \bar{X})dt + \sigma_x dZ_2. \tag{3}$$

Here we use λ_x to represent the reversion rate, $\bar{X}$ is the unconditional mean of X_t, σ_x is the processes' volatility and Z_2 is another Brownian motion. Finally, correlation existbetween the two Brownian motions in S_t and X_t, denoted by ρ.

Intertemporal Demand

Using Kim and Omberg's methodology, it can be shown that under power utility, the optimal asset allocation for an investor with risk adversion γ is

$$\theta^*(X_t, \tau) = \left(\frac{X_t}{\sigma\gamma} + \frac{\rho\sigma_x C(\tau)X_t + \rho\sigma_x B(\tau)}{\sigma\gamma}\right) \tag{4}$$

with the ODEs being:

$$\frac{dC}{d\tau} = cC^2(\tau) + bC(\tau) + a \tag{5}$$

$$\frac{dB}{d\tau} = cB(\tau)C(\tau) + \frac{b}{2}B(\tau) + \lambda_x\bar{X}C(\tau) \tag{6}$$

$$\frac{dA}{d\tau} = \frac{c}{2}B^2(\tau) + \frac{1}{2}\sigma_x^2 C(\tau) + \lambda_x\bar{X}B(\tau) \tag{7}$$

a, b and c are detailed in Kim and Omberg (1996). The first term in Equation (4) is the well-known formulation for the myopic demand of the risky asset (see Merton (1971) for more details). The last term of θ^* is termed

are perfectly positively correlated and exponential utility function was used instead of HARA.

intertemporal demand. It is classified as 'hedging against risk premium uncertainty' when the investor departs from myopic demand in order to increase wealth when risk premium becomes less favourable. Similarly, 'speculating against risk premium uncertainty' is defined as a departure from the myopic demand in order to increase wealth when risk premium becomes more favourable. Investors who have a risk-aversion,γ >1 always hedge, while for investors with γ <1, they always speculate.

From Equation (4), one can see that this intertemporal demand exists when the correlation between the two Wiener random variables is not zero. When it is zero, the optimal allocation is simply the myopic demand as noted by Merton (1971). This is because with no correlation at all, the investor cannot predict the direction of the shocks to the risk premium with respect to the shocks to the risky asset process itself, thus cannot exploit any opportunities. Given that the investor can rebalance his/her portfolio every period, the investor would just look at the current opportunity set and invest according to the myopic demand. When new information arrives in the next period (that is, changes to the instantaneous return of the risky asset), the investor will readjust his/her investment portfolio accordingly.

This is not the case when there is a correlation between the Wiener processes. As there is a relationship between the two processes, investors can predict the direction of the Wiener processes with respect to each other. The predictability allows investors to exploit this relationship to their advantage. For a investor with γ greater than 1, he/she would hedge against the risk, while investors with γ less than 1 will speculate. To illustrate, when the risk premium is positive and the correlation is negative, a investor that hedges (with a γ greater than 1) will hold more of the risky asset. This is because the risky asset will have a higher return when investment opportunities are poor. Therefore, the investor will increase his/her allocation to the risky asset so if there is a drop in investment opportunities, he/she would have a higher wealth than in the myopic case.

The direction of the intertemporal demand (that is, whether the investor would hold more or less of the risky asset) depends on three factors: $1-\gamma$, ρ and $X - X_*$ where X_* is the risk premium at which non-myopic and myopic investors behave indifferently. The results are tabulated in Table 2.1 (Kim and Omberg, 1996).

When X_t is sufficiently negative, the investor that hedges would short the asset. This is expected since the investor is better off by shorting stocks and invest the proceeds into the risk free asset. What is surprising is that with predictability, the investor would actually short at a magnitude greater than the myopic demand (a negative hedging amount). The rationale for this is that it is the magnitude of the risk premium, not the sign, which deter-

Table 1 M/L — whether the investor will hold more (M) or less (L) than the myopic demand level. Similarly, H/S — hedges or speculates.

Summary of Results for M/L and H/S from Kim and Omberg

$1-\gamma$	ρ	$X - X_*$	M/L	H/S
+	+	+	M	S
+	-	+	L	S
+	+	-	L	S
+	-	-	M	S
-	+	+	L	H
-	-	+	M	H
-	+	-	M	H
-	-	-	L	H

mines the current attractiveness of the asset. When the risk premium is low, movements toward 0 actually represent a deteoriation in the investment opportunities as shorting becomes more expensive. Therefore, to hedge against it, the investor shorts more than the myopic demand so his/her wealth will be higher if X_t does move towards 0.

This is in line with the results derived by Barberis (2000). Barberis uses dividend yield to predict the returns of the stock where there is a negative correlation between the stock returns and the dividend yield. He shows that this predictability increases the asset allocation for an investor with $\gamma > 1$.

Time Dependent Intertemporal Demand

Further analysis of equation (4) reveals that the intertemporal hedging/ speculative demand is time dependent. With power utility, Wu (p.240) shows that $C(\tau)$,$C'(\tau)$, $B(\tau)$ and $B'(\tau)$ are negative when $\gamma > 1$. This indicates that as τ increases (that is, when the investment horizon increases), $C(\tau)$ and $B(\tau)$ will get more negative but increase in magnitude, consequently increasing the intertemporal demand. Thus, the investor would invest more than the myopic demand when he/she is young and gradually reduce his/her position over time. Just before the investment horizon ends, his/her allocation would simply be the myopic demand. This is what is known as age-phasing.

This result is interesting in that previously, authors such as Samuelson (1969), Merton and Richard have discovered age-phasing does not occur in complete market settings and have criticised financial planners for promoting the age-phasing investment strategies. Age-phasing seems only to exist when market is incomplete, as noted by Bodie, Merton and Samuelson (1992) in their paper on non-tradeable income. However, under this mean-reverting model, even when the market is complete, age-phasing still occurs. In fact, the investor actually holds the highest proportion in the risky asset when the market is complete (that is, when ρ is -1). Therefore, Kim and Omberg's paper seems to support the age-phasing investment strategy in today's market. This is because intertemporal demand decreases as τ decreases and assuming that the risk premium is constant, the overall allocation to the risky asset would decrease.

2.2 Jumps

Wu (2003) extends Kim and Omberg by adding jumps to the risky asset. As a result the return of the risky asset P changes to

$$\frac{dP}{P} = (\alpha_t - \lambda g)dt + \sigma dZ + (e^q - 1)dQ, \tag{8}$$

where Q is a Poisson process with parameter λ, q follows a normal distribution with mean μ_q and volatility σ_q^2 and $g = E(e^q - 1)$. Equation (8) implies that the risk premium is

$$x_t = \frac{\alpha_t - \lambda g - r}{\sigma}. \tag{9}$$

Interrelationship between jumps, myopic and intertemporal hedging

Wu has derived an approximate solution for the terminal wealth case which has the form:

$$\theta = \frac{X_t}{\sigma\gamma} + \frac{\sigma_x \rho(B(\tau) + \tau C(\tau))}{\sigma\gamma} + \frac{\lambda \hat{g}}{\sigma\gamma} \tag{10}$$

where $\hat{g} = E_t[1 + \theta^*(X_t, t)(e^q - 1)]^{-\gamma}(e^q - 1)$, capturing the marginal utility of wealth change conditional on one jump occurring. The last term is what Wu termed as 'jump demand'. On first inspection, one may believe that the optimal weight is driven by three independent sources, myopic, intertemporal and jump. However, further analysis would reveal that the jump demand is intrinsically intertwined with the other two. A minor relationship is that jumps affect the risk premium, assuming α_t to be fixed. Therefore, the myopic and intertemporal demand would inherently be adjusted to reflect the new risk premium. A more significant impact is that since $\hat{g}$ is in the jump demand it is also a function of θ^*, its effect depends on the overall position of the portfolio. As such, the myopic and intertemporal demand influences the jump demand by their contribution to θ.

2.3 Intertemporal Consumption

Recently, many researchers have mainly concentrated on deriving the optimal investment results with an objective function which aims to maximise terminal wealth terminal wealth case (Wu, Liu et al. and Henderson). From an investment perspective (for example, a fund manager), this is perfectly fine since the sole objective is to maximise wealth.

Consumption is, however, an important economic aspect in life; one of the main drivers of investing is to fund consumption over lifetime. We are interested in consumption since in the standard optimal investment and consumption problem we described in Chapter 1, the problem is formulated using utility from lifetime consumption as the building block of its objective function. Optimal consumption problems are naturally more complex to solve and will demonstrated, consumption does have a material impact on the optimal allocation and thus should not be neglected

We focus on Wachter's findings in this section.

Weighted Average Formula

Wachter extends the Kim and Omberg model by the inclusion of consumption to the model. Using a martingale approach, Wachter derived the solution to the model to be:

$$\theta = \frac{1}{\gamma}\left(\frac{\mu_t - r}{\sigma^2} - \frac{\sigma_x \int_0^{T-t} H(X_t,\rho)(A_1(\rho)X_t + A_2(\tau))d\tau}{\gamma\sigma \int_0^{T-t} H(X_t,\tau)d\tau}\right) \quad (11)$$

$$\text{where } H(X_t,\tau) = exp\left\{\frac{1}{\gamma}(A_1(\tau)X_t^2/2 + A_2(\tau)X_t + A_3(\tau))\right\} \quad (12)$$

$$A_1(\tau) = \frac{1-\gamma}{\gamma}\frac{2(1-e^{-\Delta\tau})}{2\Delta - (b_2+\Delta)(1-e^{-\Delta\tau})} \quad (13)$$

$$A_2(\tau) = \frac{1-\gamma}{\gamma}\frac{4\lambda_x\bar{X}(1-e^{-\Delta\tau})^2}{\theta(2\Delta - (b_2+\Delta)(1-e^{-\Delta\tau}))}, \quad (14)$$

and $A_3(\tau)$ is an ODE that comprises of the integral of A_1 and A_2, Δ is the discriminant, $b_2 - 4b_1b_3$, as described in Wachter (p.70).

In this model, a more meaningful explanation can be given to the last term in (11), the intertemporal hedging demand. It can be considered as a weighted-average, where $H(X_t,\tau)$ are being used as the weights. In fact, θ itself is a weighted average. This can be verified by rewritting the above expression (11) as:

$$\theta = \int_0^{T-t} \frac{H(X_t,\tau)}{\int_0^{T-t} H(X_t,\tau')d\tau'} \times \left[\frac{1}{\gamma}\left(\frac{\mu_t - r}{\sigma^2}\right) - \frac{\sigma_x}{\gamma\sigma}(A_1(\tau)X_t + A_2(\tau))\right] d\tau \tag{15}$$

Since the myopic demand does not depend on τ, the integral for the myopic demand in (15) disappears as a result.

To provide an intuitive interpretation as to why the optimal allocation is actually a weighted-average, it is necessary to understand what $H(\tau)$ represents and why it is used as the weight. In terms of $H(\tau)$, wealth can be rewritten as

$$W_t = c_t \int_0^{T-t} H(X_t,\tau)d\tau$$

Thus, $H(X_t,\tau)$ is the value of consumption in τ periods, scaled by today's consumption.

With $H(\tau)$ defined, it is now possible to explain the intuition behind the weighted-average characteristics of the optimal allocation (11). Given, as mentioned previously, that the investor treats each consumption event separately, in order to obtain the optimal allocation, the investor first treats each 'coupon' as a terminal wealth problem. Subsequently, the investor would perform a weighted average of each individual allocation and the most logical value to use as weight is $H(\tau)$. This is because it represents how much portion of wealth the 'coupon' consumes.

Finally, the allocation under terminal wealth is simply a subset of this model. In the terminal wealth case, there is no consumption during the investment horizon. Hence, $H(X_t,\tau) = 0$ for $\tau > 0$. The integral in the hedging demand term falls out and the terminal wealth result is obtained.

Impact of Consumption on Investment horizon

A new and interesting result is the impact of intertemporal consumption on the optimal asset allocation. Previously, Merton and Richard, under complete markets, have shown that analytically, the optimal asset allocation does not depend on consumption. Recently, Purcal and Wang demonstrate numerically that this is also the case in various incomplete settings. However, in Wachter's paper, she shows that for a consuming investor with a thirty-year investment horizon, the investor's allocation of wealth to the risky asset is actually less than its terminal-wealth counterpart. More accurately, it is slightly less than that of a terminal-wealth investor with an investment horizon of ten years. This indicates that with consumption, an investor views his/her horizon as just 10 years even though the actual time horizon is 30. The deviation of the optimal allocation from the terminal wealth case is due to the correlation between the risky asset and risk premium.

From an economic perspective, the term $-\frac{\int_0^{T-t} H(X_t,\rho)(A_1(\tau)X_t+A_2(\tau))d\tau}{\int_0^{T-t} H(X_t,\tau)d\tau}$ is actually a measure of the sensitivity of consumption stream with respect to X_t. It measures the percentage change of the wealth when there is a small change in the price of risk. This is similar to the duration of a bond which measures the sensivity of the price with respect to interest rate change. Thus, the hedging demand represents the 'duration' of the investor's consumption coupons with respect to X_t.

With this intuition, the phenomenon described above can be easily explained. Since there are consumption 'coupons' during the horizon, the 'duration' would obviously be less than that of having just one coupon at the terminal date. In addition, the hedging demand increases as investor's horizon increases because by adding consumption at the end, it pulls the overall duration up.

2.4 Summary

We summarise the salient points of the literature explored above as follows:

- Correlation between the risky asset and the price of risk induces intertemporal hedging/speculative demand. A negative(positive) correlation will impact positively(negatively) for an individual with a relative risk aversion ,γ of greater(less) than 1
- The jump impact depends on the overall position on the risky asset
- Consumption will reduce the magnitude of the intertemporal demand

Wachter's model is based on a restrictive assumption, that is, the market must be complete. The impact of consumption on the optimal allocation is unresolved when the market is incomplete, especially in a market that is susceptible to jumps. This forms the basis of our model presented in the next section.

3 Model

This section sets up the mean-reverting model that we will be investigating. We will first describe the market and the objective of the investor. This is followed by the establishment of the optimal control problem we wish to solve. Finally, the first-order conditions for the controls are subsequently derived.

Our model builds on the work of Wu and Wachter, discussed in Section 2.2 and 2.3. We integrate the main features of both models into a mean-reverting risk premium environment — jumps from Wu and consumption from Wachter. In addition, the model also includes the bequest and insurance elements proposed by Richard (1975). The inclusion of insurance is because individuals not only invest and consume, but also purchase life insurance over their life cycle to their loved ones.

3.1 Structure

In our model, the market consists of two assets — a risk-free asset which has a rate of return r and a risky asset, S_t. The risky asset return is mean-reverting as well as being exposed to event risk (market crash or boom) which causes abrupt changes to the price of the asset. As a result, event risks are modelled by a Poisson jump process. The rate of return of S_t follows the jump-diffusion process

$$\frac{dS}{S} = (\alpha_t - \lambda g)dt + \sigma dZ_1 + (e^q - 1)dQ, \tag{16}$$

where α_t is the instantaneous expected return, σ^2 is the stock's volatility, Q is a Poisson process with jump intensity parameter λ, q follows a normal distribution with mean η_q and volatility σ_q^2, Z_1 is a standard, one dimensional Brownian motion and g is the expectation of the jump magnitude given a jump occurred. That is, $g = E[e^q - 1]$. There is only one state variable in the model, the risk premium, X_t, which describes the states of the world. X_t obeys a mean-reverting Ornstein-Uhlenbeck process

$$X_t = -\lambda_x(X_t - \bar{X})dt + \sigma_x dZ_2. \tag{17}$$

Here we use λ_x to represent the reversion rate, $\bar{X}$ is the unconditional mean of X_t, σ_x is the processes' volatility and Z_2 is another Brownian motion. The risk premium is linked to the risky asset's instantaneous expected return by

$$X_t = \frac{\alpha_t - \lambda g - r}{\sigma}. \tag{18}$$

Finally, the correlation between the two Brownian motions in S_t and X_t is given by

$$E[dZ_1 dZ_2] = \rho dt. \tag{19}$$

An investor in this environment is modelled to consume, at a rate of c_t per annum over his/her lifetime and have a bequest motive. In order to fulfil the bequest motive, insurance at a rate P_t is purchased. He/she also forms an investment portfolio consisting of the two assets to generate a return to sustain consumption. The investment portfolio is rebalanced continuously with no transaction costs. Consequently, the investor is subject to the following wealth constraint:

$$dW = rWdt + \left(\frac{dS}{S} - r\right)\theta W dt - c_t dt - P_t dt, \tag{20}$$

where θ is the proportion of wealth allocated to the risky asset.

The formulation of the insurance premium is simply taken from Richard (1975). The level of insurance that is purchased is related to the bequest amount, denoted by Z_t, which is the amount to be paid to dependents in the event that the investor dies. It is calculated based on the equivalence principle (Bower et al. 2000). Thus, P_t can be expressed as:

$$P_t = \mu_t(Z_t - W), \tag{21}$$

μ_t being the force of mortality. It is only the excess of Z_t over W that requires the protection of insurance. There are no constraints on P_t meaning that it could be negative — this may sound strange, as it implies insurance companies *pay* investors when bequest amounts are less than wealth. However, in actual fact, because the investor already has sufficient funds to satisfy his/her current consumption as well as the bequest amount (that is, he/she is saturated), he/she is selling the excess wealth to the insurance company. In the event that the investor dies instantaneously, the insurance company will get the excess wealth. Thus, the insurance company is now 'buying' this life insurance contract rather than selling, and the situation of $P_t < 0$ can be realistically viewed an an annuitisation of the investor's excess wealth.

Using (17),(18) and (21), (20) becomes:

$$dW = rWdt + \theta W(\sigma X_t dt + \sigma dZ_1 + (e^q - 1)dQ) - c_t dt - \mu_t(W - Z_t)dt \tag{22}$$

The investor's objective is to maximise his/her expected utility over at time t subject to the budget constraint (22).

The investor solves

$$\max_{C_t, Z_t, \theta_t} E_0'[\int_0^T V(C(s), s)ds + B(Z_T, T)] \tag{23}$$

where E_0' is the conditional expectation operator over all paths of the state variables given.[4] $V(C(s), s)$ is the utility of the investor consuming $C(s)$

[4] It is condition on information availabe at time 0.

at time s, and $B(Z_T, T)$ is the utility associated with bequest Z_T with T being the future lifetime of the investor. Equation (23) formally states our stochastic control problem.

The controls for this problem are θ_t, c_t and Z_t. They are called controls as they are determined by the investor. Wealth and risk premium are the state variables of this problem. The investor has no direct control over the state variables. For wealth, the investor can indirectly control over it through his/her choice of the controls.

3.2 Utility of consumption and bequest

Similar to many papers in optimal control (Purcal 2003, Wu, and Wachter), the power function will be used as the investor's utility function, defined as

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma} \tag{24}$$

where γ is the relative risk aversion of the investor.

The power utility function has the advantage that it is homothetic. Homotheticity implies that the investor has a constant relative risk aversion (CRRA) so he/she would behave indifferently with different levels of wealth. This simplifies the numerical analysis in later sections as the optimal weight obtained from a particular wealth level is applicable to all levels of wealth.

With relation to the forms of $V(C(s), s)$ and $B(Z_T, T)$, they are directly taken from Richard (1975). They are

$$V(C(s), s) = h(s)U(C(s)) = h(s)\frac{C(s)^{1-\gamma}}{1-\gamma} \text{ and}$$

$$B(Z_T, T) = m(T)U(Z_T) = m(T)\frac{Z_T^{1-\gamma}}{1-\gamma}.$$

Richard did not specify $h(s)$ and $m(t)$ directly, however, it is common practice (Purcal and Wachter) to set $h(s)$ to $e^{-\beta t}$. $e^{-\beta t}$ can be considered as the discount factor with a time preference of β. For $m(t)$, we adopt Purcal's choice of

$$m(t) = e^{-\beta t}\left(\frac{2}{3}\int_t^\omega \exp(-r(\omega - t)dt\right)^\gamma, \tag{25}$$

$$\text{which then implies } Z(t) = \frac{2}{3}C_t^*\int_t^\omega \exp(-r(\omega - t))dt. \tag{26}$$

$Z(t)$ gives the present value of a certain annuity from t to terminal age ω where the coupon payments are two-thirds of his/her current consumption.

To facilitate presentation, we set

$$\phi(t) = \frac{2}{3}\int_t^{\omega} \exp(-r(\omega - t))dt.$$

3.3 Hamiliton-Jacobi-Bellman (HJB) Equation

Having established the environment and market, the standard stochastic control approach is now applied to the control problem (see Merton 1971 for a detailed explanation). Following the standard procedures, we define the indirect utility function as

$$J(W, X_t, t) = \max_{\theta_s, t \leq s \leq T} E_t \left[\int_t^{\omega} \, {}_T p_t \mu_{t+T} \left[\int_t^T V(C_s, s)ds + B(Z_T, T) \right] dT \right] \tag{27}$$

The terms ${}_T p_t$ and μ_{t+T} are the standard probability of survival and force of mortality respectively (Bowers). Rather than obtaining the normal HJB equation, the current HJB equation is derived instead (Purcal). The advantage of the current HJB is that the values are larger so when applying numerical techniques, the values obtained are more accurate. Consequently, we redefine the indirect utility function to

$$J(W, X_t, t) = \max_{\theta_s, t \leq s \leq T} E_t \left[e^{\beta t} \int_t^{\omega} \, {}_s p_t e^{-\beta s} \left(\mu_s \phi(s)^{\gamma} \frac{Z(T)^{1-\gamma}}{1-\gamma} + \frac{C(s)^{1-\gamma}}{1-\gamma} ds \right) \right] \tag{28}$$

with boundary condition

$$J(W, X_{\omega}, \omega) = \phi(\omega)B(\omega)$$

Using Ito's jump-diffusion lemma, the current HJB equation is

$$0 = \max \left\{ \mathcal{L}J + J_t - (\mu_t + \beta)J + \mu_t \phi^{\gamma} \frac{Z^{1-\gamma}}{1-\gamma} + \frac{C^{1-\gamma}}{1-\gamma} \right\} \tag{29}$$

where $\mathcal{L}$ is known as the Dynkin operator; it is applied here to J over the variables W and X_t, yielding

$$\begin{aligned} \mathcal{L}J \;=\; & \lambda E_t[J(W', X_t, t) - J(W, X_t, t)] + J_W(rW + \theta\sigma X_t W - c_t - Z_t \mu_t + \mu_t W) \\ & + \frac{1}{2} J_{WW}(\theta\sigma W)^2 - J_X \lambda_x (X_t - \bar{X}) + \frac{1}{2} J_{XX} \sigma_x^2 + \rho\theta W \sigma\sigma_x J_{WX} \end{aligned} \tag{30}$$

where $W' = W[1 + \theta(e^q - 1)]$ is the wealth of the investor conditional of a jump occuring and the notation J_y represents the partial derivative of J with respect to y.

Rewriting Equation (30), we have

$$\mathcal{L}J = \lambda E_t[J(W', X_t, t) - J(W, X_t, t)] + J_W d^+ - J_W d^- + \frac{1}{2} J_{WW}(\theta\sigma W)^2 + J_x f^+ - J_x f^- + \frac{1}{2} J_{XX}\sigma_x^2 + \rho\theta W\sigma\sigma_x J_{WX} \tag{31}$$

with

$$d^+ = rW + \theta\sigma X_t W + \mu_t W \tag{32}$$
$$d^- = c_t + Z\mu_t \tag{33}$$
$$f^+ = \lambda_x \bar{X} \tag{34}$$
$$f^- = \lambda_x x_t \tag{35}$$

The reason for such notation will become clear in the next section.

Furthermore, the first-order conditions for the controls are:

$$\theta(W, X_t, t) = \left(\frac{-J_W}{J_{WW}W}\right)\frac{x_t}{\sigma} + \left(\frac{-J_{WX_t}}{J_{WW}W}\right)\frac{\sigma_x\rho}{\sigma} + \frac{\lambda E_t[J_{W'}(W', x_t, t)(e^q - 1)]}{-J_{WW}W\sigma^2} \tag{36}$$
$$c_t = U_{c_t}^{-1}(J_W) \tag{37}$$
$$Z_t = U_Z^{-1}\left(\frac{J_W}{\phi(t)^\gamma}\right) \tag{38}$$

The notation F^{-1} represents the inverse of the function F.

These first-order conditions specify the optimal values for the controls. Hence, to calculate the optimal θ_t, the investor would apply J to Equation (36).

The first term on the right hand side of (36) is the standard formulation for myopic demand for the risky asset. The subsequent term represents the investor's intertemporal demand arising from the correlation of the risk premium and the risky asset. The final term is the jump demand, as classified by Wu, which is discussed in Section (2.2).

Based on Wachter's study, it seems that a closed form solution does not exist when the market is incomplete.[5] As such, we proceed with a numerical approach.

4 Numerical implementation

This section details how the model developed in section 3 is implemented numerically. We begin by explaining the numerical and discretisation schemes

[5] See Wachter (2004) for discussion.

that are used. These schemes are then applied to our model to arrive at a set of linear equations which required to be solved. The process of solving this set of linear equations is, thus, the subject of the next section. This section concludes by specifying the parameters to be used for the model.

4.1 The numerical scheme

The finite-difference method is utilised in this paper. It essentially discretises the continuous control problem into a discrete version and uses the discrete values to approximate the derivatives. This method has been widely used in the literature. For example, Brennan et al. uses the finite-difference method to calculate the strategic asset allocation to stocks, long-term bonds and risk-free asset.

In terms of the type of scheme adopted, we chose the implicit scheme over the explicit scheme. The implicit scheme uses values from the previous grid of state variables as well as values from the current grid. Since the values in the current grid are what we wish to calculate, a system of linear equations is derived and has to be solved simultaneously. This obviously is much more computational intensive than the explicit scheme. However, the explicit scheme is restricted by the CFL condition (Courant, Friedrichs and Lewy) (Morton and Mayers, 1994) which limits the magnitude of the time step with respect to the step size of other states. Failure to comply with this condition makes the scheme unstable.[6] The implicit scheme, however, is generally unconditionally stable[7] (Morton and Mayers, 1994). Consequently, the implicit scheme does not need a step-size as small as its explicit counterpart.

There are three state spaces in this model, namely wealth, risk premium and time. In our modelling, wealth ranges from 0 to 20 and is discretised into $P+1$ grid points. Therefore, the step size is $20/P$. For risk premium, it ranges from 0 to twice the value of $\bar{X}$ with a grid of $Q+1$ points. Finally, time has a step-size of 0.1. For our scheme, we denote the discrete version of $J(W, x_t, t)$ as $J^t_{h,i}$ where t represents time, $0 \le h \le P$ and $0 \le i \le Q$.

Based on Kushner and Dupuis (1992), the following implicit scheme is used:

$$\frac{\partial J}{\partial t} = \frac{J^{t+\Delta t}_{h,i} - J^t_{h,i}}{\Delta t} \tag{39}$$

[6]Stability refers to the global error of the numerical scheme. A scheme is said to be stable if the global error has an upper bound.

[7]The implicit scheme would still have to satisfy the CFL condition but the structure of the scheme ensures that the condition is satisfied in most scenarios

$$\frac{\partial J}{\partial W} = \frac{J^t_{h+1,i} - J^t_{h,i}}{\Delta W} \text{ for } d^+ \tag{40}$$

$$\frac{\partial J}{\partial W} = \frac{J^t_{h,i} - J^t_{h-1,i}}{\Delta W} \text{ for } d^- \tag{41}$$

$$\frac{\partial^2 J}{\partial W^2} = \frac{J^t_{h+1,i} - 2J^t_{h,i} + J^t_{h-1,i}}{2\Delta W} \tag{42}$$

$$\frac{\partial J}{\partial x} = \frac{J^t_{h,i+1} - J^t_{h,i}}{\Delta X} \text{ for } f^+ \tag{43}$$

$$\frac{\partial J}{\partial x} = \frac{J^t_{h,i} - J^t_{h,i-1}}{\Delta W} \text{ for } f^- \tag{44}$$

$$\frac{\partial^2 J}{\partial x^2} = \frac{J^t_{h,i+1} - 2J^t_{h,i} + J^t_{h,i-1}}{2\Delta X} \tag{45}$$

For the mixed partial derivative, a first-order accurate implicit scheme is used:

$$\frac{\partial^2 J}{\partial W \partial X} = \frac{J^t_{h+1,i+1} - J^t_{h,i+1} - J^t_{h+1,i} + J^t_{h,i}}{\Delta W \Delta X}. \tag{46}$$

Substituting these into the HJB equation, the discretised version becomes

$$\begin{aligned} 0 = \max_{c_t, Z_t, \theta} \Bigg\{ & \mu(t)\phi(t)^\gamma U(Z_t) + U(c_t) - \mu(t)J - \beta J + \frac{J^{t+\Delta t}_{h,i} - J^t_{h,i}}{\Delta t} \\ & + \lambda E_t[J(W', X_t, t) - J(W, X_t, t)] + \frac{J^t_{h+1,i} - J^t_{h,i}}{\Delta W}(rW + \theta\sigma x_t W + \mu W) \\ & - \frac{J^t_{h,i} - J^t_{h-1,i}}{\Delta W}(c_t + Z_t\mu) + \frac{1}{2}\frac{J^t_{h+1,i} - 2J^t_{h,i} + J^t_{h-1,i}}{2\Delta W}(\theta\sigma W)^2 \\ & - \frac{J^t_{h,i-1} - J^t_{h,i}}{\Delta W}\lambda_x X_t + \frac{J^t_{h,i+1} - J^t_{h,i}}{\Delta W}\lambda_x \bar{X} \\ & + \frac{1}{2}\frac{J^t_{h,i+1} - 2J^t_{h,i} + J^t_{P,Q-1}}{2\Delta W}\sigma_x^2 + \rho\theta W\sigma\sigma_x \frac{J^t_{h+1,i+1} - J^t_{h,i+1} - J^t_{h+1,i} + J^t_{h,i}}{\Delta W \Delta X} \Bigg\} \end{aligned} \tag{47}$$

Moving the J^t terms to the left of the equation, the like terms are collected together and the coefficients are detailed in Table 2:

We assume that $E_t[J(W', X_t, t) - J(W, X_t, t)]$ is predetermined, hence considered as a constant. Section (4.3) will detail how this is actually determined.

Table 2 Coefficient for the J terms in the HJB.

Terms	Coefficients
Constants	$\Delta t(\mu_t \phi(t)^\gamma U(Z(t)) + U(c_t) + \lambda E_t[J(W', X_t, t) - J(W, X_t, t)])$
$J_{h,i}^{t+1}$	1
$J_{h,i}^{t}$	$1 + \mu\Delta t + \beta\Delta t + (\sigma\theta X_t W + rW + \mu W)\frac{\Delta t}{\Delta W} + (c_t + \mu_t Z)\frac{\Delta t}{\Delta W} + (\theta\sigma W)^2 \frac{\Delta t}{\Delta W^2}$ $+\lambda_x(X_t + \mu)\frac{\Delta T}{\Delta X} + \sigma_x^2 \frac{\Delta t}{\Delta x^2} - \theta W \sigma_x \sigma \rho \frac{\Delta t}{\Delta X \Delta W}$
$J_{h+1,i}^{t}$	$(\frac{(\theta\sigma W)^2}{2} + (\theta\sigma X_t W + rW + \mu W)\frac{\Delta T}{\Delta W} + \theta W \sigma_x \sigma \rho \frac{\Delta T}{(\Delta X \Delta W)}$
$J_{h-1,i}^{t}$	$-(\frac{(\theta\sigma W)^2}{2} + (c_t + \mu Z)\Delta W)\frac{\Delta T}{\Delta W^2}$
$J_{h,i-1}^{t}$	$-(\frac{\lambda_x X_t}{\delta X} + \frac{\sigma_x^2}{(2\Delta X^2)})\Delta T$
$J_{h,i+1}^{t}$	$(-\frac{\sigma_x^2}{(2\delta X^2)} - \frac{\lambda_x \mu}{\Delta X})\Delta T + \frac{\theta W \sigma_x \sigma \rho \Delta T}{(\Delta X \Delta W)}$
$J_{P+1,Q+1}^{t}$	$-\theta W \sigma_x \sigma \rho \frac{\Delta T}{\Delta X \Delta W}$

Similarly, the discretise version of all the controls are:

$$\theta(W, X_t, t) = -2\frac{(J_{h,i}^{t+\Delta t} - J_{h,i}^{t})}{(J_{h+1,i}^{t} - 2J_{h,i}^{t} + J_{h-1,i}^{t})}\frac{X_t}{W\sigma} - 2\frac{(J_{h+1,i+1}^{t} - J_{h,i+1}^{t} - J_{h+1,i}^{t} + J_{h,i}^{t})}{\Delta X(J_{h+1,i}^{t} - 2J_{h,i}^{t} + J_{h-1,i}^{t})}\frac{\sigma_x \rho}{\sigma} - \frac{2\Delta W \lambda E_t[J_{W'}(W', x_t, t)(e^q - 1)]}{(J_{h+1,i}^{t} - 2J_{h,i}^{t} + J_{h-1,i}^{t})W\sigma^2} \quad (48)$$

$$c_t = (\frac{J_{h,i}^{t} - J_{h-1,i}^{t}}{\Delta W})^{-\frac{1}{\gamma}} \quad (49)$$

Z_t does not need a discrete version since we can obtain it from c_t and $\phi(t)$ where the latter term does not depend on the value function.

4.2 Boundary Conditions

Equation (47) represents an equation for $J_{h,i}^{t}$. There are $(P+1)\times(Q+1)$ grid points in our grid so there would be $(P+1)\times(Q+1)$ equations. However, there are in fact more than $(P+1)\times(Q+1)$ unknowns. This is because equations for the boundaries of the grid are actually referring to J which is outside our specified. For instance, the equation where $h = P$ and $i = Q$ refers to $J_{P+1,Q+1}^{t}$. However, the point $(P+1, Q+1)$ does not exists in our finite grid. Consequently, assumptions are needed for values beyond the boundaries. The following boundary conditions are made:

$$J_{P+1,j} = J_{P,j} \tag{50}$$

$$J_{i,-1} = J_{i,0} \tag{51}$$

$$J_{i,Q+1} = J_{i,Q} \tag{52}$$

These conditions are similar to the reflecting boundary conditions suggested by Kushner and Dupuis (1992). Furthermore, there is one more boundary that needs to be addressed — when wealth is zero. In that state, the investor is bankrupt and the power utility states that the utility is negative infinity. Obviously, it is impossible to input negative infinity to represent this and so the utility is set at -10^{20} for all t when wealth is zero.

With these boundaries conditions, there are now only $P \times (Q+1)$ unknown but there are $(P+1) \times (Q+1)$ equations so a solution can be found.

4.3 Method of solving implicit scheme

If the coefficients listed in Table 2 are all known, then J^t can be found be performing a matrix inversion on the set of equations. Unfortunately, the controls θ_t , c_t and Z_t are functions of J^t itself as well. Furthermore, the function θ_t also is a function of itself. To deal with these non-linear functional relationships, the standard policy iterative approach is used (Fitzpatrick and Fleming 1990). The process is:

1. An initial guess is made of θ_t, c_t and Z_t by using $J^{t+\Delta t}$.[8] For example, to obtain θ_t, the $J^{t+\Delta t}$ values are substituted into (48). Also (48) requires J^t, however, at this stage, we assume that $J^{t+\Delta t} = J^t$.

2. Using this θ_t, the jump demand (the third term on the right in (48)) is calculated and subsequently, the θ is recalculated. Due to the complexity in deriving J values with the term W', it warrants its own subsection.

3. Step 2 is repeated twice. This is to minimise the difference between the θ_t on the left and the θ_t in the jump demand term. (Actual testing shows that repeating the process three times yields good convergence.)

4. By using θ_t, c_t and Z_t obtained from previous steps, the coefficients described in (2) is calculated and set up as a matrix such that the product of the matrix and a vector of $J_{h,i}$ yields the set of equations in (47). With these values, the value function J^t can be computed by a simple matrix inversion.

5. Using this new J^t, Steps 1-4 are repeated 2 more times but for each iteration, the new J^t is used as the value function.

[8]In the case of terminal age, the myopic demand is used.

6. The J^t is subsequently considered as the approximation to the true value of J^t

4.4 Determining the Jump Demand

The terms $E[J(W', X_t)]$ and $E[J_{W'}(W', X_t)(e^q - 1)]$ are complicated to determine as it is in reference to W' rather than W. The J value functions that is currently available cannot be directly applied since they are only in terms of W and X_t. Hence, we need J values for W'. An added complexity that jumps follows a particular distribution, namely log-normal. The key to calculate these values is to first transform the current wealth, W, to the new wealth W'.

We will first assume that q is constant and θ is available.[9] When this is the case, W' can be determined by the expression $W(1 + \theta(e^q - 1)$. With W', $J(\text{W'}, X_t, t)$, which we denote by J', is derived by referencing the corresponding values in $J(W, X_t, t)$. This is because J is only a function of wealth, risk premium and time. Jumps have no direct impact on J. As an example, let investor's wealth decreases from 10 to 5 after a jump occurs. The J' value for this investor after the jump is now $J(5, X_t, t)$. When W' is between two points on the grid, linear interpolation is used. Boundary conditions are applied for wealth outside the grid.

Another transformation that is necessary is $W' - \Delta W$. This transformation is required to perform the partial derivative $J_{W'}(W', X_t, t)$. The J' values one grid point below W', that is $J'^t_{h-1,i}$, cannot be used as the step size between $W'_{h,i}$ and $W'_{h-1,i}$ may not be ΔW. Moreover, the steps sizes in W' are not the same. This is because although the relative impact to each wealth level is the same, however the absolute impact is different. To illustrate, suppose 4 and 5 are 2 adjacent wealth on the current $J(W, X_t)$ grid and assume θ is 0.5[10] and $e^q - 1 = 0.9$. After a jump, wealth level 4 becomes 2.2 while wealth 5 becomes 2.75 so the wealth difference in W' is now 0.5 instead of 1. This difference is different when you choose 2 other adjacent wealth. Hence, $J(W' - \Delta W, X_t)$ needs to be determined separately. With $J(W', X_t)$ and $J(W' - \Delta W, X_t)$, $J'_W(W', X_t)$ can be calculated.

This is only the case when q is constant. What happens when q follows a distribution as described in our model. Fortunately, the distribution case is simply an extension to the constant q case. Initially, the jump size q is discretised into M intervals where the range is three standard deviations on either side of the mean. We denote each individual jump size as q_i. The

[9] As explained in Section (4.3), θ is a guess of the correct θ.

[10] θ is the same for both cases since CRRA utility is used

probability associated for q_i is derived by summing the density over $q_i - \frac{\Delta q}{2}$ and $\frac{\Delta q}{2}$.[11] For q_i at the boundaries, it also includes all densities passing the boundaries. Subsequently, for each q_i, $J(W', X_t)$ and $J_{W'}(W', X_t)(e^q - 1)$ are determined. Finally, we simply find $E[J(W', X_t)]$ and $E[J_{W'}(W', X_t)(e^q - 1)]$ by performing an expectation over the q_is.

4.5 Errors

There are two sources of error present in this implicit scheme. The first is discretisation error. Since we are using discrete values to approximate the derivatives, there will always be inherent errors in our results. However, as an implicit rather than explicit scheme is used, this error is bounded. To reduce the error, the step size needs to be reduced. Given that the implicit scheme is only first-order accurate in time, wealth and risk premium, in order to reduce the error by half, the step size needs to be halved for all states (time, wealth and risk premium). This would dramatically increase the computational time required since we have three state variables.

The second source of errors comes from the boundary conditions. As it is impossible to calculate the boundary conditions for $t \neq T$, appropriate guesses were made to the boundaries. Hence, errors are introduced into the system. These boundary conditions are fairly inaccurate — we can see, for the case of power utility, θ_t should be the same for all levels of wealth but due to the boundary conditions imposed, $\theta_t = 0$ when wealth is at its maximum (20). However, with the small step size that is chosen, the impact of this error is limited.

4.6 Setting

The Japanese economy has been chosen in order to study the effects of jumps and mean-reversion. Japan has been chosen as values obtained from using these parameters can be easily compared with results from previous studies (for instance, Purcal and Wang). As mean reversion and jump parameters are unavailable, they were obtained from other studies. For mean reversion, parameters were taken from Wachter (2003) since the risk premium for Japan, 0.1, is close to the risk premium in Wachter's paper. With relation to the jump parameters, it is assumed that the mean jump size is -0.05 which represents approximately a 5% drop in market value with a standard deviation of 0.04. The negative skewness is due to the fact that abrupt drops in market are more frequent than increases. For the jump frequency parameter,

[11] This is done numerically performed by using the cumulative distribution function.

we observe values from other studies. Wang uses a value of 0.1 as empirical studies suggest a major crash happening every ten years. However, in our model, we also take into account positve jumps as well so it increases the likelihood of an 'event' occurring. Furthermore, our mean jump size is less than what Wang uses. Based on these difference, we set λ to 0.2.

The parameters are summarised in Table 3.

Table 3 Parameters for our model.

Parameter	Value
$\bar{X}$	0.1
σ	0.2
r	0.005
ρ	-0.9424
β	0.005
λ_x	0.0226
σ_x	0.0189
λ	0.2
η_q	-0.05
σ_q	0.04

5 Results

This section presents the results derived from our model using parameters as specified in Section 4.6. The numerical scheme describe in section 4 is implemented in MATLAB on a Pentium PC.

5.1 Constant investent opportunities with no jump

We begin by removing the mean-reverting process and the jump component from the model we described in Section 3, arriving at the Richard's model. This is achieved by setting λ_x, λ and σ_x to 0. As proved by Richard, the optimal allocation weight for this particular is the myopic demand. This is tabled in 4. [12]

The results are comparable to Wang (2004) Table 6.4 (p.72). A 0.1 year time step was used to produce these results. Under an explicit scheme,

[12] Since the price of risk does not move between states, the model is essentially solving for a system of equations for each independent price of risk. Consequently, only 3 price of risks are used to speed up convergence.

Table 4 Richard's model. Parameters are derived from Wang (2004) which is the Japanese economy. $x = 0.1$, $r = 0.005$ and $\sigma = 0.2$

Age	$\gamma = 1.5$			$\gamma = 5.0$		
	Risky investment	Consumption	Bequest	Risky investment	Consumption	Bequest
30	33.08%	0.1781	7.5110	9.80%	0.1740	7.4263
40	33.11%	0.2022	7.6790	9.80%	0.1740	7.5939
50	33.13%	0.2357	7.8935	9.81%	0.2311	7.8084
60	33.15%	0.2841	8.1603	9.82%	0.2791	8.0752
70	33.16%	0.3583	8.4757	9.82%	0.3526	8.3910
80	33.17%	0.4836	8.8369	9.83%	0.4768	8.7530
90	33.17%	0.7253	9.1031	9.84%	0.7163	9.0719
100	33.17%	1.4016	9.0645	9.84%	1.3837	8.9619

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however, a time step of this magnitude would cause instability after 5 time step. To achieve stability, the explicit scheme requires a time step in the order of 0.0001. Thus, the implicit scheme achieves the same accuracy with a larger time step (about 1000 times) than the explicit scheme. This demonstrates the advantage of the implicit scheme.

Unfortunately, when we try to apply this program in modelling terminal utility, the program fails after several iterations. This is because under the terminal utility case, that is, without consumption and bequest, the HJB equation becomes

$$0 = \max_{\theta} J_t + J_W + \frac{1}{2} J_{WW}$$

where one of the trivial solution is when $J_t = J_W = J_{WW} = 0$, that is when all the J values are the same. By setting the step size for wealth too small, the J values are more closer together and thus converges to the same value more easily. Increasing the step size would delay the failure.

5.2 Constant investment opportunities with constant jumps

Only a minor adjustment needs to be made to the previous model for this test. The values of λ and η_q are set to values extracted from Wang (2004)[14]. To ensure that the results are accurate, Wu's results will be used calculate the theoretical value[15]. Table 5 displays the results.

[14] Wang simply used 0.1 as the magnitude of the jump size rather than using the exponential function. As such, his values were converted to indices.

[15] Following Wu's suggestion, we assume that θ is 0 initially and θ is calculated recursively. Only a few iterations are required to yield satisfactory values

Table 5 Wang's model. Parameters are the same as Table 4. In addition, $q = -0.10536$

Jump Frequency	Theoretical Risky Investment (Wu)	Risky Investment (Our model)	Consumption	Bequest
0.01	31.58%	31.36%	0.1777	7.4888
0.05	24.68%	24.53%	0.1757	7.4067
0.10	16.25%	16.15%	0.1738	7.3318

The values obtained in this study is again comparable to Wang's and more importantly, are very close to the theoretical values. This confirms that Wang's results are accurate. Similar to Wang's findings, the consumption and bequest are only marginally impacted by jumps.

5.3 Multiple roots

A peculiarity observed by Wang in his study was the instability of the optimal weight when wealth is towards the boundary, that is when wealth is close to 20. The optimal weights fluctates as seen from Figure 6.2 Wang (2004). The cause of this effect was not known. Wang investigated the possibility of multiple roots or the non-smoothness of the value function. However, none of those were fruitful and so the cause of this instability is still unknown.

Using the implicit iterative method, however, seems to resolve this issue, as shown in Figure 1. The differences between this study's implementation and Wang's are the method in deriving θ and the finite difference scheme used. In order to obtain θ, Wang solves the first-order condition by finding the root to the equation. Our study uses an iterative approach and an implicit scheme rather than an explicit one. Given these differences and that Wang was unable to find multiple roots, it suggests that the instability is caused by the explicit scheme. Thus a reduction in time step or an increase in the wealth interval may resolve the issue.

5.4 Numerical Problems

While testing, however, it was observed that when x_t is 0 and λ or η is large in magnitude, a similar effect occurs. For instance, when $\lambda = 0.5$ and $\eta = -0.10536$, the values for theta is shown in Figure 2: As one can observe, there is a large unusual spike. The reason for this phenomenon is that when $x_t = 0$, shorting occurs. Consequently, the jump demand, $E_t[1 + \theta(e^\eta - 1)]^{-\alpha}(e^\eta - 1)$ is greater than 1. Due to the boundary condition that is imposed, the partial, $J_{W'}$ becomes 0 when wealth is close to maximum. As such, the value function becomes unsmooth since some value function

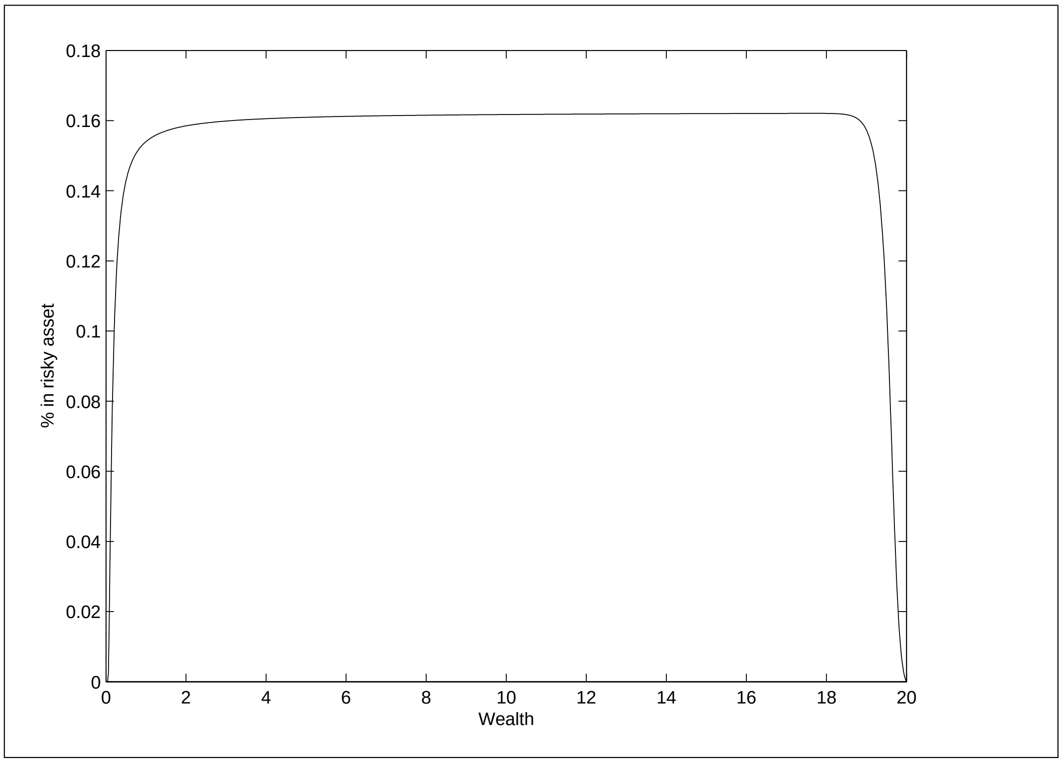

Figure 1: Optimal allocation (%) against wealth using Wang's parameters.

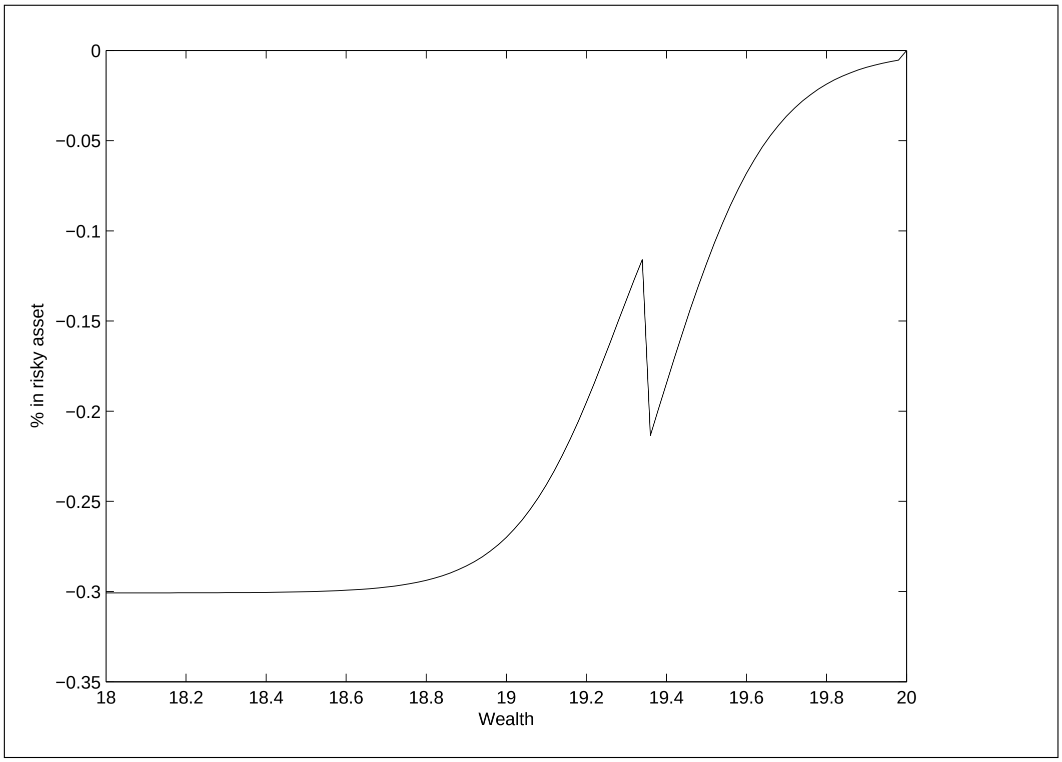

Figure 2: Optimal allocation (%) against wealth when $\lambda = 0.5$.

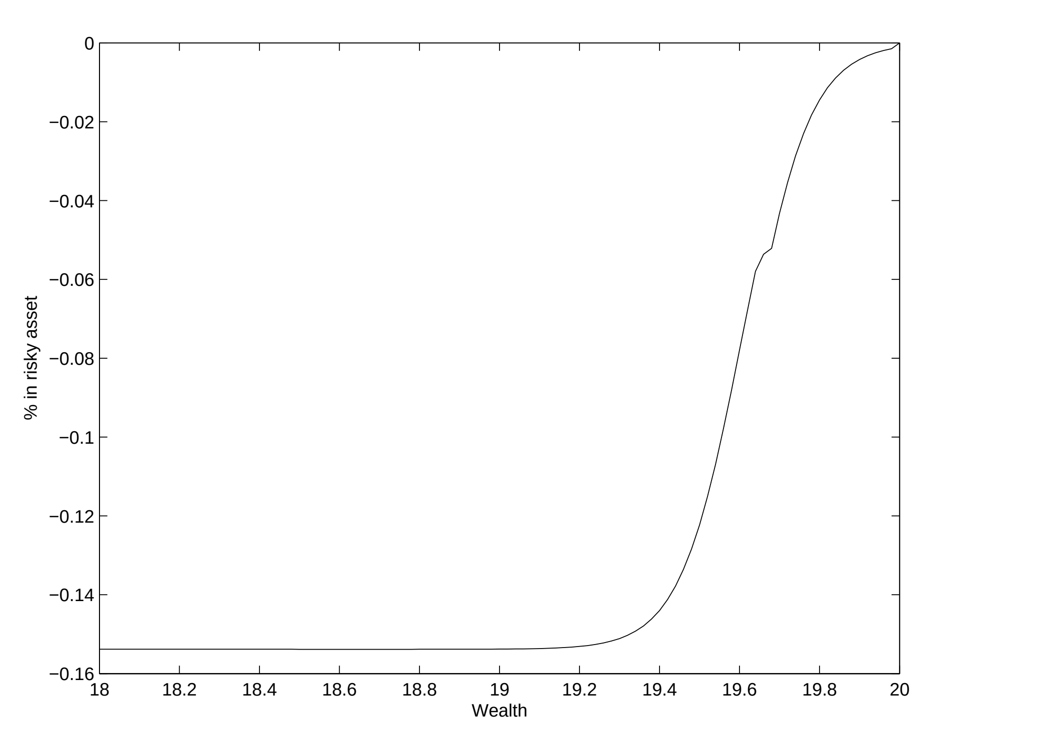

Figure 3: Optimal allocation (%) against wealth when $\lambda = 0.1$.

contain the jump demand while others don't. This is not a problem when only a small portion of wealth are affected. To demonstrate, we adjust λ to be 0.2, so the shorting is reduced, translating to less wealth points being over the boundary. Figure 3 displays the function θ over time:

There is still a kink in the θ but it is not as sharp as the kink in Figure 2. As a result, this kink is smoothed out after going back several time-steps. This becomes a problem when too many values do not contain the jump demand component.

This is unrelated to the problem discussed in the last section as in Wang's case, the market only experiences crashes. Therefore, given that the risk premium is positive, no shorting would occur. Thus this problem would not occur in Wang's model.

5.5 Mean-reverting investment opportunities with jumps but no correlation

Table 5.5 displays the results when there is no correlation but jumps exist.

To dissect the two demand, we first calculate the standard myopic de-

Mean-reversion with jumps but no correlation			
	γ=1.5		
Age	Myopic Demand	Jump Demand	Overall Demand
30	32.61%	-15.82%	16.79%
40	32.73%	-15.92%	16.81%
50	32.83%	-16.01%	16.82%
60	32.91%	-16.08%	16.83%
70	32.96%	-16.12%	16.84%
80	32.99%	-16.14%	16.85%
90	33.01%	-16.15%	16.86%
100	33.01%	-16.15%	16.86%

mand, which can be obtained from Section 3. The overall optimal allocation, including jumps, is then obtained from our model[16]. Finally, the jump demand is simply the difference between the two values. As it clearly demonstrates, mean-reversion without correlation does not have any impact on the jump demand. The jump demand is relatively stable. It is 16.86% when the investor is aged 100 and decreases to 16.79% at the age of 30. While there is a slight decrease in the value, however, the reduction is by no means near the magnitude observed by Wu, as reported in Section ...[17]. The differences observed in this result are mainly due to the discretisation error introduced by the numerical scheme.

5.6 Mean reversion with jumps and correlation

The results of the two controls, θ and c_t, are shown in Figures 6 and 7. With a correlation very close to -1, we find that, in the absence of jumps, the intertemporal hedging demand is not as large as observed by other studies including Campbell and Viceria and Wachter. Wachter reported that the intertemporal hedging demand is approximately 50% of the myopic demand even when risk premium is 0.1 (the risk premium for our study) and $\gamma = 4$ for a 30 year period. However, over a 80 year period, our model suggests that it is only a mere 10% of the myopic demand. The reason for such a dramatic difference is due to the extreme relative sizes of the volatility of the risky asset and the risk premium. σ is 0.2 while σ_x is only 0.0189, which is 10 times smaller. As a result, in order to hedge against changes in the risk premium, the investor will only need to increase his/her allocation to the

[16] X_t is not adjusted as suggested by Wu

[17] Wu does use different values but tests on different parameters do not contradict the nature of our observation

Table 6 Optimal theta. The optimal asset allocation is split into two parts: myopic (θ_m) and intertemporal (θ_h). θ without subscript s has jump risk taken into consideration.

Age	Myopic		Intertemporal Hedging		Overall	
	θ_{ms}	θ_m	θ_{hs}	θ_h	θ_s	θ
30	32.61%	16.79%	3.36%	1.68%	35.97%	18.47%
40	32.73%	16.81%	3.21%	1.69%	35.94%	18.50%
50	32.83%	16.82%	3.06%	1.76%	35.89%	18.58%
60	32.91%	16.83%	2.90%	1.84%	35.81%	18.67%
70	32.96%	16.84%	2.69%	1.88%	35.65%	18.72%
80	32.99%	16.85%	2.35%	1.81%	35.34%	18.66%
90	33.01%	16.86%	1.73%	1.42%	34.74%	18.28%
100	33.01%	16.86%	0.90%	0.73%	33.91%	17.59%

Table 7 Optimal consumption. The optimal consumption is split into two parts: myopic (c_m) and intertemporal (c_h). c without subscript s has jump risk taken into consideration.

Age	Myopic		Intertemporal Hedging		Overall	
	c_{ms}	c_m	c_{hs}	c_h	c_s	c
30	0.1781	0.1746	0.0008	0.0003	0.1789	0.1749
40	0.2022	0.1989	0.0008	0.0003	0.2030	0.1992
50	0.2357	0.2326	0.0006	0.0003	0.2363	0.2329
60	0.2841	0.2813	0.0005	0.0002	0.2846	0.2815
70	0.3583	0.3558	0.0003	0.0003	0.3586	0.3561
80	0.4836	0.4817	0.0004	0.0002	0.4840	0.4819
90	0.7253	0.7245	0.0009	0.0001	0.7262	0.7246
100	1.2387	1.4072	0.1703	0.0000	1.4090	1.4072

risky asset marginally. Having a large intertemporal demand would actually make the investor worse off, since a small increase in X_t will mean a large drop in the return of the risky asset. As expected, the correlation increases the consumption of the investor with or without jumps. The reason for the marginal improvement could be explained by the low relative risk adversion we have set. With a γ of only 1.5, the investor is not very risk adverse, thus prefers to invest rather than consume.

6 Conclusion

In this paper, we have examined the optimal behaviour of an investor in Wu's model of mean-reverting risk premium with predictability and jump risks. By optimal behaviour, we refer to the decisions made by the investor in choosing consumption and allocation to the risky asset in order to maximise lifetime utility. Comparisons were made with previous studies to validate their findings.

As noted by Kim and Omberg, we have found that predictability creates non-myopic behaviour for a risk-averse investor. The investor will hold more/less than the myopic in order to hedge/speculate future investment opportunities. When consumption is considered, the non-myopic impact is lessened. This is in line with Wachter who has derive a closed-form solution for the optimal allocation when intertemporal consumption part of the investor's behaviour. As expected, jumps reduced the overall allocation to the risky asset but only marginally impacting the consumption behaviour of the investor. We were, however, unable to observe a decrease in myopic demand when jumps and predictability interact.

In this study, the finite difference method is used on PDE to approximate the value functions and subsequently, computing the optimal controls. However, this method does have its limitation. One of the concerns is the accuracy of the values obtained due to errors discussed in 4.5. The results in Section 5.2 shows that there is about 1% difference between the theoretical value and the numerical value.

A more concerning problem is that the finite difference scheme does not work for all different parameters. As demonstrated in Section (5.4), when X_t is low and λg is high, our model fails due to the finiteness of the state space available. This indicates that this method cannot be applied to all scenarios, limiting its practial use.

References

[1] Balduzzi, P. and Lynch, A. W. (1999), Transaction Costs and Predictability: Some Utility Cost Calculations, Journal of Financial Economics, 52, 47-48.

[2] Barberis, N. (2000), Investing for the Long Run When Returns are Predictable, Journal of Finance, 55, 225-264.

[3] Bowers, N., Gerber, H., Hickman, J., Jones, D. and Nesbitt, C. (1997), Actuarial Mathematics, Society of Actuaries.

[4] Brennan, M., Schwartz, E. and Lagnado, R. (1997), Strategic Asset Allocation, Journal of Economic Dynamics and Control, 21, 1377-1403.

[5] Campbell, J. and Shiller, R. (1988), The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors, Review of Financial Studies, 1, 195-227.

[6] Campbell, J. and Viceira, L. (1999), Consumption and Portfolio Decisions When Expected Returns are Time-varying, Quarterly Journal of Economics, 114, 433-495.

[7] Chacko, G. and Viceira L. (2000), Dynamic Consumption and Portfolio Choice With Stochastic Volatility in Incomplete Markets, Working paper, Harvard University.

[8] Cont, R. and Tankov, P. (2004) Financial Modelling With Jump Processes, Chapman & Hall/CRC Financial Mathematics Series, 2004.

[9] Duffie, D., Pan, J. and Singleton, K. (2000), Transform Analysis and Asset Pricing for Affine Jump Diffusion, Econometrica, 68, 1343-1376.

[10] Fama, E. F. and French K. R. (1989), Business Conditions and Expected Returns on Stocks and Bonds, Journal of Financial Economics, 29, 23-49.

[11] Heston, S. L. (1993), A Closed-form Solution for Options With Stochastic Volatility With Applications to Bond and Currency Options, Review of Financial Studies, 6, 328-341.

[12] Kim, T. and Omberg, E. (1996), Dynamic Nonmyopic Portfolio Behavior, Review of Financial Studies, 9, 141-161.

[13] Kushner, H. J. and Dupuis, P. (2001), Numerical Methods for Stochastic Control Problems in Continuous Time, Applications of Mathematics, Stochastic Modelling and Applied Probability 24, 2nd edn, Springer-Verlag.

[14] Liu, J., Longstaff, F. and Pan, J. (2002), Dynamic Asset Allocation With Event Risk, Journal of Finance, 55, 2003.

[15] Luenberger, D. G. (1998), Investment Science, Oxford University Press.

[16] Merton, R. C. (1969), Lifetime Portfolio Selection Under Uncertainty: The Continuous-time Case, Review of Economics and Statistics, 51, 247-257.

[17] Merton, R. C. (1971), Optimum Consumption and Portfolio Rules in a Continuous-time Model, Journal of Economic Theory, 3, 373-413.

[18] Merton, R. C. (1973), An Intertemporal Capital Asset Pricing Model, Econometrica, 41, 867-869.

[19] Merton, R. C. (1976), Option Pricing When the Underlying Stock Returns are Discontinuous, Journal of Financial Economics, 3, 125-144.

[20] Merton, R. C. (1990), Continuous-time Finance, Basil Blackwell.

[21] Poterba, J. M. and Summers, L. H. (1988), Mean Reversion in Stock Prices: Evidence and Implications, Journal of Financial Economics, 22, 27-59.

[22] Purcal, S. (2003), Growing Old Gracefully: Optimal Financial Behaviour Over a Stochastic Life-cycle, PhD thesis, University of New South Wales.

[23] Wachter, X. (2002), Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets, Journal of Financial and Quantitative Analysis, 37-1, 63-91.

[24] Wu, L. (2003), Jumps and Dynamic Portfolio Decisions, Review of Quantitative Finance and Accounting, 20, 207-243.

[25] Xia, Y. (2001), Learning About Predictability: The Effect of Parameter Uncertainty on Dynamic Asset Allocation, Journal of Finance, 56, 205-246.