Valuation of Long Term Equity Options and Guarantees under Stochastic Interest Rates

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Outline

1. Long Term Guarantees and Interest Rate Variability
2. HJM framework and applicable models
3. Simulation methods: Issues and Tools

Long Term Contracts

- Examples:
  - Executive Share Option Plans
  - Embedded Equity Return Guarantees
- Standard Approach: Black-Scholes framework
  - Theory well understood (mathematical finance)
  - Numerical methods widely available
  - Deterministic Term Structure

Example: European Call

- Guarantee: Earn at least current risk free rate
- Stock: Geometric Brownian Motion, initial price 100.
- Compare Prices under:
  - constant risk free rate
  - Ho-Lee stochastic term structure

Example: European Call

Term (years)	Constant	Ho-Lee
10	25.3	28.9
0.25	4.0	4.0

Importance of Interest Rate Variability

- May be significant for long term contracts
- Typically insignificant for short term contracts
- Also depends on
  - Parameters
  - Guarantee Type
- Trade-off between
  - Sophisticated Modeling (Computation Effort)
  - Significance to Results

Theory: Derivative Pricing

- Standard Pricing Theory:

$$E_Q\left[e^{-\int_0^T r(t)dt} \left(S(T)-K\right)^+ \right]$$

- Heath-Jarrow-Morton framework
  - Current Term Structure as input
  - Consider forward rate curve f(t,T)
  - Apply No arbitrage condition

HJM – forward rates

- The Q dynamics of the forward rate f(t,T) is

$$df(t,T) = \sigma(t,T)\left(\int_t^T \sigma(t,s)ds\right)dt + \sigma(t,T)dW(t)$$

- short rate: r(t)=f(t,t)
- The volatility functions selected via
  - Historical Volatility Curve
  - Ease of Implementation
  - (Fit to actively traded interest rate derivatives)

HJM - Assets

- Savings Account

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}$$

- ZCB with maturity T

$$dZ(t,T) = Z(t,T)\left(r(t) + \frac{1}{2}\varphi^2(t,T)\right)dt + Z(t,T)\varphi(t,T)dW$$

$$\varphi(t,T) = -\int_t^T \sigma(t,u)du$$

- Stock

$$dS(t) = r(t)S(t)dt + \upsilon S(t)dW$$

HJM – short rate models

- These will give good candidate models for our purposes.
- Ho-Lee

$$\sigma(t,T)=\phi$$

$$dr=\theta(t)dt+\phi dW$$

  – Brownian Motion with drift under Q, Gaussian

HJM – short rate examples

- Vasicek

$$dr(t) = \alpha\left(\kappa(t) - r(t)\right)dt + \phi dW$$

  – Mean Reversion, Gaussian

- Cox-Ingersoll Ross

$$dr(t) = \alpha\left(\kappa(t) - r(t)\right)dt + \phi\sqrt{r(t)}dW$$

  – Strictly Positive (Non-Central Chi-Squared)

HJM - Calibration

- Calibration to current term structure.
- Could involve extrapolating yield curve
- For many pricing models, only need
  - calibration of integrals $\int_{t_i}^{t_{i+1}} \theta(u)du$
  - calibration to a fixed/small number of dates

Simulation Model – First Approach

- Theoretical Model:

$$d \ln S(t) = r(t)\,dt - \frac{1}{2}\upsilon^2 dt + \upsilon d\tilde{W}$$

$$dr = \theta(t)dt + \phi(t)\,dW$$

- Need Increments of process

$$\ln S(t_{i+1}) = \ln S(t_i) + \int_{t_i}^{t_{i+1}} r(u)du - \int_{t_i}^{t_{i+1}} \frac{1}{2}\upsilon^2 du + \int_{t_i}^{t_{i+1}} \upsilon d\tilde{W}$$

$$r(t_{i+1}) = r(t_i) + \int_{t_i}^{t_{i+1}} \theta(u)du + \int_{t_i}^{t_{i+1}} \phi(t)\,dW$$

Euler Scheme

- Discrete Approximation

$$\ln S(t_{i+1}) = \ln S(t_i) + r(t_i)(t_{i+1} - t_i) - \frac{1}{2}\upsilon^2 (t_{i+1} - t_i) + \upsilon\left(\tilde{W}(t_{i+1}) - \tilde{W}(t_i)\right)$$

$$r(t_{i+1}) = r(t_i) + \theta(t_i)(t_{i+1} - t_i) + \phi(t_i)\left(W(t_{i+1}) - W(t_i)\right)$$

- Requires only Gaussian random variables
- Flexible

Discretization and Simulation

- Discretization: additional Error
- Mean Squared Error = Variance + Bias
- Trade-off:
  - Computational Effort
  - Small MSE

Managing (Simulation) Variance

- Variance Reduction Methods
  - Control Variates
  - Stratified Sampling
  - Importance Sampling
  - Combination
- Best method generally problem-specific
- For current setup, CV can be very efficient

Managing Variance - Control Variates

- Couple payoff of interest with a related variable whose expected value is known
- New Estimator

$$\overline{Y}(b) = \overline{Y} - b\left(\overline{X} - E[X]\right)$$

- Optimal coefficient

$$b = \frac{Cov(X,Y)}{Var(X)}$$

- Ratio of Variance Reduction

$$1 - Corr(X,Y)^2$$

Managing Variance - Control Variates

- Very useful CV – Option with no interest rate variability
- Same Brownian increments used, but with interest rate variability turned off
- Typically the CV has either
  - Closed form solution
  - Very efficient numerical procedures (simulation or otherwise)
- Eg: Call Option: 95% correlation

Managing Bias

- Simple Euler scheme can have significant bias.
- Finer Time-steps:
  – Lower bias
  – Significant computation effort (long term contracts)

Managing Bias – Richardson Extrapolation

- With time steps of size h

$$E\left[f\left(S_h\right)\right]=E\left[f\left(S\right)\right]+ch+o\left(h\right)$$

- With time steps of size 2h

$$E\left[f\left(S_{2h}\right)\right]=E\left[f\left(S\right)\right]+c2h+o(h)$$

- Combining:

$$2E\left[f\left(S_h\right)\right]-E\left[f\left(S_{2h}\right)\right]=E\left[f\left(S\right)\right]+o\left(h\right)$$

Managing Bias – Richardson Extrapolation

- Method is simple to implement
- Correlation between $f(S_h)$, $f(S_{2h})$ important for minimal variance
- Can use the same Brownian increments for the 2 estimators

Managing Bias – Exact r

- The current simulation setup requires discretization of both
  - r(t)
  - Integral of r(t)
- With some models it is possible to simulate r exactly
  - Ho-Lee, Vasicek: Gaussian
  - CIR: non-central Chi-Squared

Managing Bias - Exact integral

- For Gaussian models it is possible to simulate directly the integral of r(t)
- Sum of normals
- Involve 0 discretization error

Managing Bias – a note on Correlation

- Correlation between stock returns and interest rate variability can complicate procedures
- Note:
  - Instantaneous Correlation
  - Effect of Correlation of Final output often negligible

Conclusions

- Interest Rate Variability can be important for some long term guarantees
- Tradeoff between sophistication and computation effort
- Consider model choice, and methods of
  - Variance Reduction
  - Bias Reduction
- Solution will be Problem-Specific

Extension: Stochastic Volatility, Jumps

- Alternative stock price models can be allowed for in the simulation approach.
- Stochastic Volatility – Heston model
- Euler Scheme with extrapolation
- CV – some SV models have essentially closed form prices
- Jumps: Poisson jumps can be allowed for

Reference and Support

- Main references:
  - Glasserman (2004): Monte Carlo Methods in Financial Engineering
  - James and Webber (2000): Interest Rate Modelling
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